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THE

ELEMENTS OF ALGEBRA;

DESIGNED FOR

THE USE OF SCHOOLS.

BY THE

REV. J. W. COLENSO, D.D.,

BISHOP OF NATAL.

160

OS. 1925

PART I.

FROM THE THIRTEENTH LONDON EDITION.

NEW YORK:

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In this Edition (which is *stereotyped*, and so will be secured from further change) the Simpler Parts, those, namely, suited for general School purposes and required for the attainment of an ordinary B.A. degree in the University of Cambridge, are printed separately as Part I; to which is appended a large collection of easy Miscellaneous Examples, specially adapted to the contents of this Part, and supplying means of complete Examination in them.

It will be seen that the easiest kinds of Simple Equations and Equation Problems are in this Edition introduced much earlier than is usual in Treatises on Algebra: but there can be no reason why this branch of the subject, which is so interesting to most Students, and gives them some idea of the practical applications of the Science, should not be brought forward as soon as possible.

Part II is also published, and contains the higher parts of the Subject, with such additional remarks on the earlier portions as will suit the wants of more advanced and promising Students, and with a similar Appendix of more difficult Miscellaneous Examples and Equation Papers. This Part may be begun as soon as the Student, having thoroughly mastered Part I, has entered upon the Miscellaneous Examples at the end of it.

Forncett St. Mary, Nov. 1, 1849.

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ALGEBRA.

PART I.

CHAPTER I.

DEFINITIONS.

1. ALGEBRA is the science which reasons about quantities by means of letters of the Alphabet, and certain signs and symbols, which are employed to represent both the quantities themselves, and the manner in which they are connected with others.

Thus we might put a to represent 7, and then *twice* a would represent 14; or we might put a to represent 3, and then *twice* a would represent 6, *three times* a, 9, &c.

2. The sign = (*equal*) denotes that the quantities between which it stands are equal to one another.

Thus, if $a = 17$, then *twice* $a = 34$.

3. The sign $\therefore$ stands for *then* or *therefore*, and $\because$ for *since* or *because*.

4. The sign + (*plus*) denotes that the quantity before which it stands is *added*, and the sign − (*minus*) that the quantity before which it stands is *subtracted*.

Thus $5 + 3 = 8$, $5 - 3 = 2$; and if $a = 3$ and $b = 4$,
then $a + b = 3 + 4 = 7$, $a + b + 2 = 3 + 4 + 2 = 9$,
$10 - a = 10 - 3 = 7$, $10 - a - b = 10 - 3 - 4 = 7 - 4 = 3$.

The sign $\sim$ is used to denote that the less of two quantities is taken from the greater, when it is not known which *is* the greater.

Thus $a \sim b$ denotes the *difference* between a and b.

5. All quantities before which + stands are called *positive*, and all before which − stands are called *negative* quantities.

If neither + nor − stand before a quantity, + is understood, and the quantity is positive; thus a means $+a$.

6. The sign × (*into*) denotes that the quantities between which it stands are to be multiplied together; but very often a full-point is used instead of ×, or, still more commonly, one quantity is placed close after the other without any sign between them.

Thus $a \times b$, $a.b$, and ab mean all the same thing, viz., a multiplied by b; and, therefore, if $a = 3$ and $b = 4$, we shall have $ab = 12$, $5a = 15$, $5ab = 60$; and if also $c = 5$, $d = 0$, then

$$4ab + 3ac + 4d - 2b + 2abc - 3abcd = 48 + 45 + 0 - 8 + 120 - 0$$
$$= 213 - 8 = 205.$$

7. The number, whether positive or negative, prefixed to any algebraical quantity, is called its *coefficient*; thus 3 is the coefficient of $3a$, -7 of $-7ax$, &c.

If no number is expressed, the coefficient is understood, being 1, since a means *once a*.

Ex. 1.

If $a = 6$, $b = 5$, $c = 4$, $d = 3$, $e = 2$, $f = 1$, and $g = 0$, find the numerical values of the following expressions:

1. $a + 2b + 3c + 4d + 3e + 2f + g$.
2. $2a + b - 3c + 4d - 5f + 6g$.
3. $3b - 4a - 6c + 7d + 2e - 4g$.
4. $-3a + 2b + 3c - 2e + f$.
5. $ab + 5bc - 4de + 5fg$.
6. $4ag - 3bf + 4ce - ad$.
7. $-3ab - 2ac + 4bc - abc$.
8. $5ab - 8ac + 15cde - 14aef$.
9. $33ab - 19cd + 22abg - 13cdef$.
10. $abcd - 2bcde + 3cdef - 4defg$.

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8. The sign ÷ (*by*) denotes that the quantity which stands *before* it is to be divided by that which *follows* it; but, most frequently, to express division, the quantity to be divided is placed over the other with a line between them, in the form of a fraction.

Thus $a \div b$ and $\frac{a}{b}$ denote, either of them, *a divided by b;* and if $a = 2$, $b = 3$, then

$$\frac{5a}{2b} = \frac{10}{6} = \frac{5}{3}, \quad \frac{3a + 2b}{2b - a} = \frac{6 + 6}{6 - 2} = \frac{12}{4} = 3.$$

9. When any quantity is multiplied by *itself* any number of times, the product is called a *power* of the quantity, and is briefly expressed by writing down the quantity, with a small figure above it to the right, denoting the number of times it is repeated.

Thus, a^5 stands for $a \times a \times a \times a \times a$, $3a^4b^3c^2d$ for $3aaaabbbccd$.

The small figure in any case is called the *index* of the corresponding power.

Thus, a (which means a^1) is the *first power* of a,
a^2 . . . . the *second* . . or *square* of a,
a^3 . . . . the *third* . . or *cube* of a,
a^4 . . . . the *fourth* power of a, &c., &c.,

and the small figures, ^{2}, ^{3}, ^{4}, &c., are the indices of the second, third, fourth, &c., powers of a, respectively.

Hence, if $a = 2$, $a^4 = 2 \times 2 \times 2 \times 2 = 16$,
if $a = 3$, $a^3 = 3 \times 3 \times 3 = 27$,
if $a = 1$, $a^2 = 1$, $a^3 = 1$, $a^4 = 1$, &c.

Ex. 2.

If $a = 1$, $b = 3$, $c = 5$, and $d = 0$, find the values of

1. $\frac{2b}{a} + \frac{3c}{b} + \frac{5a}{c} - \frac{2a + b}{c}$.
2. $\frac{3a + 2b}{b} - \frac{2b + 3c}{7a} + \frac{2ab - c}{a}$.
3. $\frac{ab + 2bc + 3cd}{2a + 3b} - \frac{2abc - 4ad + 3ac}{3ab - 2ad} + \frac{3abc + 6ac + 6ab - 3bc}{6c - 2b}$.
4. $a^2 + 2b^2 + 3c^2 + 4d^2$.
5. $3a^2b + 2b^2c - 2a^2c + 3b^2d$.
6. $a^3 - 3a^2c + 3ac^2 - c^3$.
7. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
8. $4abc^2 - 3a^2bc + \frac{2ab^2c}{2a + b + c}$.
9. $\frac{12a^3 - b^3}{3a^2} + \frac{2c^2}{a + b^2} - \frac{a + b^2 + c^3}{5b^3}$.
10. $\frac{a^2b^2 + 1}{a^3 + b^2} - \frac{1 + a^2c^2}{a^2 + c^2} + \frac{4a + b^3 + b^2c^2}{b^2 + c^3} - \frac{a^2 + 2ab + b^3}{b^2 - 2bc + c^2}$.

10. The *square root* of a quantity is that quantity whose square power is equal to the given quantity.

Thus the *square root* of 9 is 3, since $3^2=9$; the *square root* of a^2 is a, of 64 is 8.

So also the *cube*, *fourth*, &c., root of a quantity is that quantity whose cube, fourth, &c. power is equal to the given one.

The symbol used to denote a root is $\sqrt{}$ (a corruption of r, the first letter of the word *radix*), which, with the proper index on the left side of it, is set before the quantity whose root is expressed.

Thus, $\sqrt[2]{a^2}=a$, $\sqrt[3]{64}=4$, $\sqrt[5]{3125}=5$, $\sqrt[2]{1}=1$, $\sqrt[3]{1}=1$, &c.

The index, however, is generally omitted in denoting the *square* root; thus, $\sqrt{x}$ is written instead of $\sqrt[2]{x}$.

Find the values of Ex. 3.

1. $\sqrt{4}+2\sqrt{25}+3\sqrt{49}-\sqrt{64}$.
2. $3\sqrt{16}-4\sqrt{36}+2\sqrt{9}-\sqrt{81}$.
3. $\sqrt[3]{8}+2\sqrt[3]{125}-4\sqrt[3]{1}+\sqrt[3]{64}$.
4. $\sqrt[4]{1}+3\sqrt[4]{16}-2\sqrt[5]{32}+3\sqrt[6]{1}$.

If $a=25$, $b=9$, $c=4$, $d=1$, find the values of

5. $\sqrt{a}+2\sqrt{b}+3\sqrt{c}+4\sqrt{d}$.
6. $\sqrt{4a}+\sqrt{9b}+\sqrt{16c}-\sqrt{25d}$.
7. $3\sqrt{a}+2\sqrt{4b}-4\sqrt{9c}+\sqrt{16d}$.
8. $\sqrt[3]{5a}+2\sqrt{3b}-\sqrt[3]{2c}+4\sqrt[3]{d}$.
9. $\sqrt{a^2}-2\sqrt[3]{b^3}+3\sqrt[4]{c^4}-4\sqrt[5]{d}$.
10. $\sqrt{bc}+3\sqrt{acd}-4\sqrt{b^2d}+\sqrt{c^2d^3}$.

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11. Algebraical quantities are said to be *like* or *unlike*, according as they contain the *same* or *different* combinations of letters.

Thus, a and $5a$, $-5a^3b$ and $7a^3b$, $3a^2bc$ and $-a^2bc$, are pairs of like quantities; a^3 aud a^2, $3ab$ and $-7a$, $3a^2b$ and $3ab^2$, of unlike quantities.

12. *Brackets*, (), { }, [ ], are employed to show that all the quantities within them are to be treated as though forming but one quantity. It is of great importance to notice carefully the effect of using them.

Thus $a-(b-c)$ is not the same as $a-b-c$; for, in this last, both b and c are subtracted, whereas in the former it is the quantity, $b-c$, which is subtracted.

Hence, if $a=4$, $b=3$, $c=1$, we have

$$a-b-c=4-3-1=0,\ a-(b-c)=4-2=2;$$
$$2a-3b+2c=8-9+2=1,\ 2a-(3b+2c)=8-11=-3;$$
$$2a+b-c=8+3-1=10,\ 2(a+b)-c=14-1=13,\ 2(a+b-c)=12.$$

Sometimes, instead of brackets, a line is used, called a *vinculum*, and drawn above the quantities that are connected; thus $a-\overline{b-c}$ is the same as $a-(b-c)$.

The line, which separates the num^r^ and den^r^ of a fraction, is also a species of vinculum, corresponding, in fact, in *Division* to the bracket in *Multiplication*.

Thus $\dfrac{a+b-c}{4}$ implies that the *whole* quantity $a+b-c$ is to be divided by 4, and might have been written $\frac{1}{4}(a+b-c)$.

Ex. 4.

If $a=0$, $b=2$, $c=4$, $d=6$, find the values of

1. $3a+(2b-c)^2+\{c^2-(2a+3b)\}+\{3c-(2a+3b)\}^2$.
2. $3b+(2c-d)^2+\{3b-(2c-d)\}^2-\{3b-(2c-d)^2\}$.
3. $2\sqrt{d-b}+3\sqrt{3d+2c-1}+4\sqrt{a+b+2c+d}$.
4. $3\sqrt[3]{2b^2-a}+2\sqrt[3]{b^2+c^2+7}-\sqrt[3]{2(b+c)^2-(b+d)^2}$.
5. $\{a+(b+c)^2-d\}\ \{(a+b)^2+(d-c)^2\}\ \{(a+b+c)^2-d\}$.

If $a=1$, $b=2$, $c=3$, $d=4$, show that the numerical values are equal,

6. Of $(b+c+d)\ (b+c-d)\ (b+d-c)\ (c+d-b)$
   and of $4b^2c^2-\{d^2-(b^2+c^2)\}^2$.
7. Of $\{d-(c-b+a)\}\ \{(d+c)-(b+a)\}$,
   and of $d^2-(c^2+b^2)+a^2+2(bc-ad)$.
8. Of $\{(b+c)-(d-a)\}^2+\{(c+d)-(b-a)\}^2+\{(b+d)-(c-a)\}^2+(b+c+d-a)^2$, and of $4(a^2+b^2+c^2+d^2)$.
9. Of $\{(a+d)-(c-b)\}\ \{(a+c+d)-b\}\ \{c-(d-a-b)\}\ (b+c+d-a)$,
   and of $4(ad+bc)^2-\{(a^2+d^2)-(b^2+c^2)\}^2$.
10. Of $d^2-(2d-c)c+\{2(d-c)+b\}\ b-\{2(d-c+b)-a\}\ a$,
    and of $\{(d-a)-(c-b)\}^2$.

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13. Those parts of an expression, which are connected by the signs + or —, that is, which are connected by *Addition* or *Subtraction*, are called its *terms*, and the expression itself is said to be *simple* or *compound*, according as it contains one or more terms.

Thus a^2, $2ab$, and $-3b^2$, are each *simple* quantities, and $a^2 + 2ab - 3b^2$ is a *compound* quantity, whose *terms* are a^2, $+2ab$, and $-3b^2$.

Those parts of an expression which are connected by *Multiplication* are called its *factors*.

Thus the factors of a^2 are a and a, those of $2ab$ are 2, a, and b, those of $-3b^2$ are -3, b, and b, or, as we should rather say, -3 and b^2, it not being usual (except where specially required for any purpose) to break up a *power* into its elementary factors. Of course we might include 1 as a factor in each case; thus, since $a^2 = 1 \times a^2$, the factors of a^2 are 1 and a^2, and so of the rest: and this will be sometimes required, as will be seen hereafter, but for the present need not be attended to.

It is very necessary that the student should learn at once to distinguish well between *terms* and *factors*.

Thus $2a + b - c$ is a *compound* quantity of three *terms*, $2a$, b, and $-c$; $2(a+b) - c$ is one of two *terms* only, $2(a+b)$ and $-c$, of which the former, $2(a+b)$, consists of two *factors*, 2 and $a+b$, the factor, $a+b$, being itself a compound quantity of two terms; and so also $2(a+b-c)$ is a *simple* quantity or *single* term, of two *factors*, 2 and $a+b-c$, of which the latter is itself a compound quantity of three terms.

Let it be observed, then, that *terms* are the quantities which make up an expression by way of *Addition* or *Subtraction*, *factors*, by way of *Multiplication*.

It may be also noticed, that it is immaterial in what order either the *terms* or the *factors* of a quantity are arranged. It is usual, however, to arrange quantities, as much as possible, in the order of the alphabet.

Thus $a - 2b + 3c$ is the same quantity as $-2b + a + 3c$, or $3c - 2b + a$, &c., and abc is the same as bac or bca; but we should prefer to write $a - 2b + 3c$, and abc, unless there were some reason, in any case, for arranging otherwise.

A quantity of *one* term is called a *monomial*, of *two* terms, a *binomial*, of *three*, a *trinomial*, &c., and, generally, of *more than two* terms, a *multinomial*.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION.

14. To add *like* algebraical quantities, add separately the positive and negative coefficients; take the difference of these two sums, prefix the sign of the greater, and annex the common letters.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$3a$	$-12bc$	$2c^2$	$3a^2+2b^2$
$-5a$	$4bc$	$-5c^2$	$4a^2-3b^2$
$-2a$	$3bc$	$-7c^2$	$-8a^2+4b^2$
$5a$	$-8bc$	$10c^2$	$5a^2-6b^2$
$6a$	$5bc$	$4c^2$	$7a^2+3b^2$
$7a$	$-8bc$	$4c^2$	$11a^2$ *

In the last example the star is used to indicate that the terms involving b^2 destroy one another.

If the quantities are *unlike*, we must add any that are like by the preceding rule, and write down the others with their proper signs.

Ex. 5.	Ex. 6.	Ex. 7.
$2a+3b-4c$	$x-2y+3z$	$2a+c+d$
$-3a+4b-c$	$-2x+3y-4z$	$-b+a+e$
$4a+7b+7c$	$3x-5y-5z$	$+c-d$
$a-b-4c$	$x+y$	$-3a-e-f$
$-5a+2b-6c$	$2y+2z$	$-2c+2d-2e$
$-a+15b-8c$	$3x-y-4z$	$-b+2d-2e-f$

Find the sum of Ex. 5.

1. $7a-3b+4c-2d+7$, $-8a+4b-6c+2d-11$, $13a+3b-5c+4d-4$, $2a-b+c+11$, $a+2d-3$.

2. $2x-3y+4z-4$, $x+2y-3z$, $-3x+2y-5z+7$, $4x-y+2z-3$, $9x-10y+11z-12$, $x+y+z$.

3. $2a^2+ab+3b^2$, $3a^2-4ab+2b^2$, $3a^2+3ab-b^2$, $12a^2-14ab-7b^2$, $3a^2-12ab+17b^2$.

4. $ax-4by+3cz$, $13ax-9by+7cz$, $-5ax+7by-14cz$, $2ax-by+cz$, $-11ax+13by-4cz$.

5. $20x^3+20x^2y-3xy^2+14y^3$, $-17x^3+14x^2y-12xy^2-3y^3$, $14x^3+17x^2y+15xy^2-5y^3$, $-12x^3-13x^2y-14xy^2-5y^3$, $12x^2y+3y^3$.

6. $2x^2 - 3xy - 4y^2$, $3xz + 2y^2 - z^2$, $x^2 - 2yz + 5z^2$, $3xy - 6xz - 3x^2$, $3xz - 2z^2 + 5yz$, $4y^2 - 3yz + 2x^2$.

7. $x^3 - 3ax^2 + 3a^2x - a^3$, $4x^3 - 5ax^2 + 6a^2x - 15a^3$, $3x^3 + 4ax^2 + 2a^2x + 6a^3$, $17x^3 + 19ax^2 - 15a^2x + 8a^3$, $-13ax^2 - 27a^2x + 18a^3$.

8. $a^3 - 2ab^2 - ac^2 + a^2b + 2a^2c + 2abc$, $-2a^2b + b^3 - 2bc^2 + 2ab^2 + 2abc + b^2c$, $-2a^2c - b^2c + c^3 + 2abc + ac^2 + 2bc^2$.

9. $3x^3 + 2y^3 + z^3 + 8yz^2$, $y^3 + 3x^2y + 2xy^2 + z^3 - 23x^2z$, $x^3 + 2xyz + 4x^2y + 12x^2z - 9y^2z + 6yz^2$, $2x^3 - 3y^3 + 4xyz - 6xy^2$, $4y^3 + z^3 + 5x^2z - 15xyz + 3y^2z - 14yz^2$, $6x^2z - 15xyz + 4xy^2 - 7x^2y + 6y^2z$.

10. $x^4 + 3xy^3 - xz^3 + x^3y + x^3z$, $3x^2y^2 + 3x^2z^2 + 3xy^2z - 3xyz^2 - 6x^2yz$, $-x^3y + y^4 - yz^3 - 3x^2y^2 + 3x^2yz$, $-3xy^3 - 3xyz^2 - 3y^3z + 3y^2z^2 - 6xy^2z$, $-x^3z + 3y^3z + z^4 + 3x^2yz - 3x^2z^2$, $3xy^2z + xz^3 - 3y^2z^2 + yz^3 + 6xyz^2$.

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15. To subtract algebraical quantities, *change their signs* and proceed as in Addition.

Thus, if we take b from a, the result will be $a-b$; but, if we take $b-c$ from a, the result will be *greater* by c than the former, since the quantity now to be subtracted is *less* by c than in the former case; hence the result required will be $a-b+c$, which is therefore the value of $a-(b-c)$, so that the quantities b, $-c$, when subtracted, become $-b$, $+c$, respectively.

Or we may reason otherwise, as follows:

(i) Since $a=a-b+b$, if we subtract $+b$ from a, the result is $a-b$, the same as if we add $-b$ to it.

(ii) Since $a=a+b-b$, if we subtract $-b$ from a, the result is $a+b$, the same as if we add $+b$ to it.

Thus if a person possesses a pounds and owes b pounds, his money in hand may be represented by $+a$ pounds, and his debt by $-b$ pounds, so that he may be said to possess $+a$ and $-b$ pounds, or, in one sum, $a-b$ pounds. Now if we *subtract* or annul his debt, that is, if we take away his negative property, $-b$ pounds, he will possess the whole positive property, $+a$ pounds, the same as if we give him $+b$ pounds to pay his debt with.

There will often, however, be no need formally to apply the above rule of *changing signs*, since the difference may be obtained at once, by taking that of the coefficients and annexing the common letters.

Thus, in Ex. 1, we may say, at once, $3x$ *from* $5x$ *leaves* $2x$, y *from* $7y$ *leaves* $6y$, $-4z$ *from* $-8z$ *leaves* $-4z$; though, of course, if we chose to apply the Rule (*change the sign of the quantity to be subtracted and proceed as in Addition*) it would equally be true that $-3x$ *added to* $+5x$, $-y$ to $+7y$, $+4z$ to $-8z$, would produce respectively, $+2x$, $+6y$, and $-4z$, as before.

	Ex. 1.	Ex. 2.	Ex. 3.
From	$5x + 7y - 8z$	$5x^2 - 2xy + 3y^2$	$-3a^2 + 4ab - 5b^2$
take	$3x + y - 4z$	$-4x^2 - 2xy + 7y^2$	$-7a^2 + 3b^2 - 2c^2$
Ans.	$2x + 6y - 4z$	$9x^2 \quad - 4y^2$	$4a^2 + 4ab - 8b^2 + 2c^2$

Ex. 6.

1. From $2a - 2b + c$ take $a + b - 2c$.
2. From $2x^2 - 3xy + y^2$ take $4x^2 + 4xy - 2y^2$.
3. From $5ax - 7by + cz$ take $ax + 2by - cz$.
4. From $7x^2 - 2x + 4$ take $2x^2 + 3x - 1$.
5. From $8a^2 - 2a + 6b^2 - 5ab + 5c^2 - 3bc + 2$ take $a^2 + a + 2b^2 + 2ab + 3c^2 + 3bc + 2$.
6. From $2x^3 - 4x^2y - 3y^2 + 6 - 2x^2 - 3xy^2 - 14y^2$ take $3x^3 + 2x^2y - y^2 - 3xy^2 + x^2 - 10y^3$.
7. From $5x^2 + 6xy - 4y^2 - 12xz - 7yz - 5z^2$ take $2x^2 - 3y^2 + 4xz - 5z^2 + 6yz - 7xy$.
8. From $3x^2 + 2xy - y^2$ take $-x^2 - 3xy + 3y^2$, and $3x^2 + 4xy - 5y^2$
9. From $a^4 - 2a^3b + 3a^2b^2 - 4ab^3 + 5b^4$ take $2ab^3 - 3a^2b^2 + 4a^2b - 5a^4$, and $3a^4 - 2a^3b + 6a^2b^2 - 2ab^3 + 3b^4$.
10. From $a^5 - 4a^3b^2 - 8a^2b^3 - 17ab^4 - 12b^5$ take $a^5 - 2a^4b - 3a^3b^2$, $2a^4b - 4a^3b^2 - 6a^2b^3$, $3a^3b^2 - 6a^2b^3 - 9ab^4$, and $4a^2b^3 - 8ab^4 - 12b^5$.

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16. Since the sign + or —, preceding a bracket, will imply (12) that the whole included quantity is to be added or subtracted, if we wish to remove the bracket, we must actually perform the operation indicated by means of it, *i. e.*, we must add or subtract the quantity in question. Of course, in the case of + preceding it, this amounts to no more than merely setting down the included terms with their proper signs, because, when a quantity is added, the signs of its terms are not altered; but in the case of — preceding a bracket,

we shall have to *change the signs* of all the included terms, since they are all to be subtracted.

Thus $+(a+b-c) = \quad a+b-c, \quad (a^2-2ab-b^2) = \quad a^2-2ab-b^2$;
but $\;-(a+b-c) = -a-b+c, \; -(a^2-2ab-b^2) = -a^2+2ab+b^2$:
so also, in the case of a double bracket, we have

$$\begin{aligned} &3a - \{(a-3c)-(2b-c)\} = \\ &3a - (a-3c)+(2b-c) = \\ &3a-a+3c+2b-c = 2a+2b+2c. \end{aligned}$$

The same remark applies also to the case of a fraction with a num^r of more than one term, whenever the line separating its num^r and den^r, and which (12) is a species of vinculum, is removed by any process.

Thus $-\dfrac{a+b-c}{4}$ [or $-\frac{1}{4}(a+b-c)$] $= -\dfrac{a}{4}-\dfrac{b}{4}+\dfrac{c}{4}$ [or $-\frac{1}{4}a-\frac{1}{4}b+\frac{1}{4}c$];
and $-\frac{1}{2}(a-b)$, when multiplied by 2, becomes $-(a-b)$, or $-a+b$.

Ex. 7.

Reduce to their simplest forms:

1. $(a-x)-(2x-a)-(2-2a)+(3-2x)-(1-x)$.
2. $(a^3-2a^2c+3ac^2)-(a^2c-2a^3+2ac^2)+(a^3-ac^2-a^2c)$.
3. $(2x^2-2y^2-z^2)-(3y^2+2x^2-z^2)-(3z^2-2y^2-x^2)$.
4. $(x^3+ax^2+a^2x)-(y^3-by^2+b^2y)+(z^3+cz^2+c^2z)-(x^3-y^3+z^3)$ $+(ax^2+by^2+cz^2)-(a^2x-b^2y+c^2z)$.
5. $a^2-(b^2-c^2)-\{b^2-(c^2-a^2)\}+\{c^2-(b^2-a^2)\}$.
6. $\{2a^2-(3ab-b^2)\}-\{a^2-(4ab+b^2)\}+\{2b^2-(a^2-ab)\}$.
7. $\{x^3+y^3-(3x^2y+3xy^2)\}-\{(x^3-3x^2y)-3xy^2-y^3)\}$.
8. $\{2x-(3y-z)\}-\{y+(2x-z)\}+\{3z-(x-2y)\}-\{2x-(y-z)\}$.
9. $1-\{1-(1-4x)\}+\{2x-(3-5x)\}-\{2-(-4+5x)\}$.
10. $\{2a-(3b+c-2d)\}-\{(2a-3b)+(c-2d)\}+\{2a-(3b+c)-2d\}$ $-\{(2a-3b+c)-2d\}$.

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17. It is often necessary not only to break up, or *resolve*, quantities contained in brackets, but also to *form* such quantities, that is, to take up in a bracket any given terms of an expression. Now, in doing this, it should be noticed that, whatever term we choose to set as *first* term within the bracket, the sign of that term will have to be placed *before* the bracket, and this

sign will of course affect all the terms we may place within the bracket. If, then, this sign should be (+), the other terms may be set down at once within the bracket with their proper signs; but if it should be (—), we shall have to *change the signs* of all these other terms, and then set them within the bracket: for the sign (—), which precedes the bracket, will influence all these signs, and have really the effect of *correcting*, as it were, the changes we have made, and will, in fact, cause the original signs to reappear, whenever we choose to resolve the bracket again.

Thus $+a-b-c$, collected in a bracket with $+a$ as *first* term, will be $+(a-b-c)$; but, with $-b$ as first term, $-(b-a+c)$, and with $-c$ as first term, $-(c-a+b)$; and now, if we resolve again these last two brackets, the sign (−), preceding each of them, will correct the changes we have made, and the quantities will be reproduced, as at first, $-b+a-c$, $-c+a-b$.

So also we might use an *inner* bracket, and write the quantity $+\{(a-b)-c\}$, or $+\{a-(b+c)\}$, or $-\{(b-a)+c\}$, or $-\{b-(a-c)\}$, &c.

Ex. 8.

Express, by brackets, taking the terms (i) *two*, (ii) *three*, together,

1. $2a-b-3c+4d-2e+3f$.
2. $-b-3c+4d-2e+3f+a$.
3. $-3c+4d-2e+3f+2a-b$.
4. $+4d-2e+3f+2a-b-3c$.
5. $-2e+3f+2a-b-3c+4d$.
6. $3f+2a-b-3c+4d-2e$.

7—12. Express the *second* answer in each of the above by using also an *inner* bracket, including in it the *latter two* of the three terms within each of the outer brackets.

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18. We have spoken hitherto only of *numerical* coefficients; but, in fact, when a quantity is composed of two or more factors, any one of them is a coefficient of the rest taken together, that is, (as the word *coefficient* implies) *makes up with* them, as a factor, the quantity in question.

Thus in $3abcx$, 3 is, as before, the coefficient of $abcx$; but $3a$ is also the coefficient of bcx, $3ab$ of cx, ax of $3bc$, &c.

Such coefficients are called *literal* coefficients, as involving algebraical *letters;* and, when any terms of a quantity contain some common factor, a bracket is often employed to collect the *other* factors, considered as its literal coefficients, into one quantity, which is set before or after the common factor.

Thus we have seen already that $3x + 2x - x = 4x$, that is, $= (3 + 2 - 1)x$; and in like manner, $ax + bx - x = (a + b - 1)x$, $2a - 4ax + 6ay = 2a(1 - 2x + 3y)$, $(a + 2b)x^2 - (2b - c)x^2 - (2c - a)x^2 = \{(a + 2b) + (2b - c) - (2c - a)\}x^2 = (2a - c)x^2$.

$$\begin{array}{lrrr} \text{Add} & (a-2p)x^3 & & -2x^2+(2c-3r)x \\ & (2p+a)x^3 & +(q-b)x^2 & -x \\ & -(p-a)x^3 & -(b+q)x^2 & -(c-1)x \\ & -x^3 & +3bx^2 & -(c-2r)x \\ \hline \textit{Ans.} & (3a-p-1)x^3 & +(b-2)x^2 & -rx \end{array}$$

$$\begin{array}{lrrr} \text{From} & ax^3 & -bx^2 & +x \\ \text{take} & -px^3 & -qx^2 & +rx \\ \hline \textit{Ans.} & (a+p)x^3 & -(b-q)x^2 & +(1-r)x \end{array}$$

The above Answers may, of course, be expressed differently, by changing the order of the terms within the brackets; thus, the second might have been written $(a + p)x^3 + (q - b)x^2 - (r - 1)x$.

On the other hand, when a bracket comes in this way before or after a single term as factor, it may be resolved, after multiplying each term of the quantity within it by the common factor.

Thus $a(b - x) - (a - y)b = (ab - ax) - (ab - by) =$
$ab - ax - ab + by = by - ax = -(ax - by)$.

Ex. 9.

1. Collect coeff^{s} in $ax^3 - bx^2 - cx - bx^3 + cx^2 - dx + cx^3 - dx^2 - ex$.
2. Add together $ax - by$, $x + y$, and $(a - 1)x - (b + 1)y$.
3. Add together $(a + c)x^2 - 3(a - b)xy + (b - c)y^2$, and $(b - c)x^2 + 2(a + b)xy + (a - b)y^2$.
4. Add together $(a + b)x + (b + c)y$ and $(a - b)x - (b - c)y$, and subtract the latter from the former.
5. Add together (i) the first two, (ii) the last two, and (iii) all four together, of $2(a + b)x + 3(b + c)y$, $-3(a - b)x + 2(a - c)y$, $-(2b + c)x + (a - 2b)y$, and $(a - 2b)x - (b + 2c)y$.

6. In (5) (i) subtract the second quantity from the first, and (ii) the fourth from the third, and (iii) add the two results together.
7. In (5) (i) subtract the third from the first, and (ii) the fourth from the second, and (iii) add the two results together.
8. In (5) (i) subtract the fourth from the first, and (ii) the third from the second, and (iii) add the two results together.

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19. To multiply two simple algebraical quantities together, multiply together respectively the numerical coefficients and letters; and then, if the multiplier and multiplicand have the *same* sign, prefix to this product the sign +, if *different* signs, the sign −.

Thus, $7a \times 4b = 28ab$, $-2a \times 3c = -6ac$, $5b \times -2c = -10bc$, $-3a \times -5b = 15ab$.

This rule for determining the sign of the product, viz. that *like signs produce* +, *and unlike* −, may be thus deduced.

Let it be required to multiply $a-b$ by $c-d$.

Here $(a-b)(c-d) = (a-b)x$, (writing x for $c-d$),

$$\begin{aligned} &= ax-bx \\ &= a(c-d)-b(c-d) \\ &= (ac-ad)-(bc-bd) \\ &= ac-ad-bc+bd: \end{aligned}$$

in which result we see that the product of $+a$ by $+c$ is ac (i. e. $+ac$), that of $+a$ by $-d$ is $-ad$, that of $-b$ by $+c$ is $-bc$, and that of $-b$ by $-d$ is $+bd$.

If several simple quantities are to be multiplied together, instead of multiplying them together successively by the above rule, (thus $2a \times -3b \times -4c = -6ab \times -4c = 24abc$), it will be shorter to multiply them at once together, and then prefix to this product the sign + or −, according as the number of *negative* factors is even or odd.

20. The powers of a quantity are multiplied together by *adding the indices.*

Thus $a^3 \times a^3 = a^{3+2} = a^5$; for $a^3 = a.a.a$, $a^2 = a.a$; $\therefore a^3 \times a^2 = a.a.a.a.a = a^5$; and so in other cases.

Hence

$-3a^2b \times 4a^3b^2 \times -2a^2b^3 = 24a^7b^6$, $2abc \times 3a^2b^2c^2 \times -ab^2c = -6a^4b^5c^4$.

21. If the multiplier or multiplicand consist of several terms, each term of the latter must be multiplied by each term of the former, and the sum of all the products taken for the complete product of the two quantities.

This process is generally conducted as in the following Examples.

Ex. 1.
$$\begin{array}{l} 3x^2 - 2xy + 4y^2 \\ 2a^2x \\ \hline 6a^2x^3 - 4a^2x^2y + 8a^2xy^2 \end{array}$$

Ex. 2.
$$\begin{array}{l} -2a^2b^2 + 5ab^3 - 7b^4 \\ -4ab \\ \hline 8a^3b^3 - 20a^2b^4 + 28ab^5 \end{array}$$

Ex. 3.
$$\begin{array}{llll} a & + b & & \\ a & + b & & \\ \hline a^2 & + ab & & \\ & + ab & + b^2 & \\ \hline a^2 & + 2ab & + b^2 & \end{array}$$

Ex. 4.
$$\begin{array}{lll} a & + b & \\ a & - b & \\ \hline a^2 & + ab & \\ & - ab & - b^2 \\ \hline a^2 & * & - b^2 \end{array}$$

Ex. 5.
$$\begin{array}{lll} a & - b & \\ a & - b & \\ \hline a^2 & - ab & \\ & - ab & + b^2 \\ \hline a^2 & - 2ab & + b^2 \end{array}$$

Ex. 6.
$$\begin{array}{lll} x & + a & \\ x & + b & \\ \hline x^2 & + ax & \\ & + bx & + ab \\ \hline \end{array}$$

Ans. $x^2 + (a+b)x + ab$

Ex. 7.
$$\begin{array}{l} x^2 + (a+b)x + ab \\ x + c \\ \hline x^3 + (a+b)x^2 + abx \\ \quad + \quad c\,x^2 + (ac+bc)x + abc \\ \hline x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc \end{array}$$

Ex. 8.
$$\begin{array}{l} x^3 - ax^2 + bx - c \\ x^2 + mx + n \\ \hline x^5 - ax^4 + bx^3 - cx^2 \\ \quad + mx^4 - amx^3 + bmx^2 - cmx \\ \qquad + nx^3 - anx^2 + bnx - cn \\ \hline \end{array}$$

Ans. $x^5 + (a-m)x^4 + (b-am+n)x^3 - (c-bm+an)x^2 - (cm-bn)x - cn$

Ex. 10.

1. Multiply ax^2y^3 by bxy; mx^2 by $-nx^3$; $-acx$ by $-2axy$; abc by bc; $-abc$ by $-ac$; x^2y by $-xy^2$.
2. Multiply $x^2 - xy + y^2$ by x, and $a^2 - ax + x^2$ by $-ax$; $x^2 - ax + b$ by $-abx$; $x^3 - 3x^2y + 3xy^2 - y^3$ by xy.
3. Multiply $2a + b$ by $a + 3b$, and $2a - b$ by $c - 3d$.
4. Multiply $3x + 2y$ by $2x + 3y$, and $3ab + 4b^2$ by $2ab - 3b^2$.
5. Multiply $x^2 + 3x - 2$ by $x + 3$, and $x^2 - 4x + 3$ by $x - 2$.
6. Multiply $a^2 + 2a - 1$ by $a^2 - a + 1$, and by $a^2 - 3a - 1$.
7. Multiply $27x^3 + 9x^2y + 3xy^2 + y^3$ by $3x - y$.
8. Multiply $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$ by $a + 2b$.
9. Multiply $x^2 + 2ax + 3a^2$ by $x^2 - 2ax + a^2$.
10. Multiply $9a^2 - 3ab + b^2 - 6a - 2b + 4$ by $3a + b + 2$.
11. Multiply $x^2 + y^2 + z^2 + xy - xz + yz$ by $x - y + z$.
12. Multiply $a^3 + 2a^2 + 2a + 1$ by $a^3 - 2a^2 + 2a - 1$.
13. Multiply $a^2 + 4b^2 + 9c^2 + 2ab + 3ac - 6bc$ by $a - 2b - 3c$.
14. Multiply $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$ by $a^2 + 2ab + b^2$.

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15. Multiply $x^2 - ax + b$ by $x - c$, and by $x^2 + ax - c$.
16. Multiply $1 - ax + bx^2 - cx^3$ by $1 + x - x^2$.
17. Multiply $a + mx - nx^2$ by $a - 2mx + nx^2$, and by $a + 2nx - mx^2$
18. Find the continued product of $ax - by$, $ax + cy$, and $ax - dy$.
19. Find the continued product of $2x-m$, $2x+n$, $x+2m$, and $x-2n$.
20. Find the continued product of $x^2 + ax - b^2$, $x^2 + bx - a^2$, and $x - (a+b)$.

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22. The student should notice some results in Mult^n, so as to be able to apply them when similar cases occur, and write down at once the corresponding products.

Thus, (21 Ex. 3, 5) the product of $a + b$ by $a + b$, or the square of $a + b$, is $a^2 + 2ab + b^2$, and the square of $a - b$ is $a^2 - 2ab + b^2$: by remembering these results, we may write down at once the square of any other binomial; thus,

$$(x + y)^2 = x^2 + 2xy + y^2,\ (x - 2)^2 = x^2 - 4x + 4,\ (2x + y)^2 = 4x^2 + 4xy + y^2,$$
$$(2ax - 3by)^2 + 4a^2x^2 - 12abxy + 9b^2y^2.$$

Again, (Ex. 4) the product of $a + b$ by $a - b$ is $a^2 - b^2$: hence we have $(x + y) \times (x - y) = x^2 - y^2$, $(x + 2)\ (x - 2) = x^2 - 4$,

$$(2ax + 3by)\ (2ax - 3by) = 4a^2x^2 - 9b^2y^2.$$

So, also, (Ex. 6) the product of $x + a$ by $x + b$ is $x^2 + (a + b)\,x + ab$, where the coeff. of x is the *sum* of the two latter terms of the factors, $x + a$, $x + b$, and the last term, $+ ab$, is their *product:* in like manner, we shall have

$$(x + 5)(x + 2) = x^2 + (5 + 2)x + 10 = x^2 + 7x + 10,$$
$$(x - 5)(x + 2) = x^2 + (2 - 5)x - 10 = x^2 - 3x - 10,$$
$$(x + 2)(x - 2)(x + 3)(x - 3) = (x^2 - 4)(x^2 - 9)$$
$$= x^4 - (9 + 4)x^2 + 36 = x^4 - 13x^2 + 36,$$
$$(x + 2)(x - 3)(x - 4)(x + 5) = (x^2 - x - 6)(x^2 + x - 20)$$
$$= (\text{by common Mult}^{\text{n}})\, x^4 - 27x^2 + 14x + 120.$$

23. Let then these three results, or *formulæ*, be noted:

$$\text{(i)}\ (a \pm b)^2 = a^2 \pm 2ab + b^2;$$

or, *the square of any binomial = the sum of the squares of its two terms together with twice their product:*

$$\text{(ii)}\ (a + b)(a - b) = a^2 - b^2;$$

or, *the product of the sum and difference of any two quantities = the difference of their squares:*

$$\text{(iii)}\ (x + a)(x + b) = x^2 + (a + b)x + ab.$$

24. By a little ingenuity, however, the above formulæ may be still more extensively applied to lighten the labor of Mult$^{\text{n}}$: thus

Ex. 1. $(a - b + c)^2 = \{(a - b) + c\}^2 =$ by (i) $(a - b)^2 + 2(a - b)c + c^2$ $= a^2 - 2ab + b^2 + 2ac - 2bc + c^2$; or we might have written it $\{a - (b - c)\}^2$, or $\{(a + c) - b\}^2$, &c., and then have expanded either of these by (i), obtaining, of course, the same result as before: but we shall give a better method hereafter for squaring a trinomial; it will be sufficient to have noticed this.

Ex. 2. $(a^2 - ax + x^2)(a^2 - ax - x^2) =$ by (ii) $(a^2 - ax)^2 - x^4$
$$= a^4 - 2a^3x + a^2x^2 - x^4.$$

Ex. 3. $(a^2 + ax - x^2)(a^2 - ax - x^2) = \{(a^2 - x^2) + ax\}\{(a^2 - x^2) - ax\}$ $= (a^2 - x^2)^2 - a^2x^2 = a^4 - 2a^2x^2 + x^4 - a^2x^2 = a^4 - 3a^2x^2 + x^4$.

Note that the formula here employed, $(a + b) \times (a - b) = a^2 - b^2$, may be always applied, whenever it is seen that the two quantities to be multiplied consist of terms, which differ only (some of them) in sign, by taking for a those terms which are found *with their signs unaltered* in each of the given quautities, and the others for b: thus, in Ex. 3, a^2 and $-x^2$ appear in both the given quantities, whereas in one we have $+ax$, in the other $-ax$; hence the product required is $(a^2 - x^2)^2 - a^2x^2$, as above.

Ex. 4. $(a^2 + ax + x^2)(a^2 - ax + x^2) = (a^2+x^2)^2 - a^2x^2 = a^4 + a^2x^2 + x^2.$

Ex. 5. $(a^2+ax-x^2)(a^2-ax+x^2) = a^4 - (ax-x^2)^2 = a^4 - a^2x^2 + 2ax^3 - x^4.$

Ex. 6. $(a^2-ax+x^2)(ax+x^2-a^2) = x^4 - (a^2-ax)^2 = x^4 - a^4 + 2a^3x - a^2x^2.$

Ex. 7. $(a + b + c + d)(a + b - c - d) = (a + b)^2 - (c + d)^2$
$= a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$

Ex. 8. $(a + 2b - 3c - d)(a - 2b + 3c - d) = (a - d)^2 - (2b - 3c)^2$
$= a^2 - 2ad + d^2 - 4b^2 + 12bc - 9c^2.$

Ex. 11.

1. Write down the squares of $a - x$, $1 + 2x^2$, $2a^2 + 3$, $3x - 4y$.
2. Write down the squares of $3 + 2x$, $2x - 3y$, $a^2 - 3ax$, $bx^2 - cxy$.
3. Write down the product of $(2a + 1) \times (2a - 1)$, $(3ax + b) \times (3ax - b)$, $(x - 1)(x + 1)(x^2 + 1)$.
4. Write down the product of $(x + 3)(x + 1)$, $(x^2 + 4)(x^2 - 1)$, $(ab - 3)(ab + 2)$, $(2ax - 3b)(2ax - b)$.
5. Find the continued product of $x + a$, $x - a$, $x + 2a$, and $x - 2a$.
6. Find the continued product of $mx + 2ny$, $mx - 2ny$, $mx - 3ny$, and $mx + 3ny$.
7. Simplify $3(a-2x)^2 + 2(a-2x)(a+2x) + (3x-a)(3x+a) - (2a-3x)^2$.
8. Multiply $x^2 + 2xy + 2y^2$ by $x^2 - 2xy + 2y^2$, and $2a^2 - 3ab + b^2$ by $2a^2 + 3ab + b^2$.
9. Multiply $a + b + c$ by $a + b - c$, by $a - b + c$, and by $a - b - c$.
10. Multiply $a - b + c$ by $a - b - c$, by $b + c - a$, and by $c - b - a$.
11. Multiply $2a + b - 3c$ by $2a - b + 3c$, and by $b + 3c - 2a$.
12. Multiply $2a - b - 3c$ by $2a + b + 3c$, and by $b - 3c - 2a$.
13. Multiply $a + b + c + d$ by $a - b + c - d$, by $a - b - c + d$, and by $b + c - b - a$.
14. Multiply $a - 2b + 3c + d$ by $a + 2b - 3c + d$, by $2b - a + 3c + d$, and by $a + 2b + 3c - d$.

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25. To divide one *simple* algebraical quantity by another, divide respectively the coefficient and letters of the dividend by those of the divisor; and then, if the two quantities have the same sign, prefix to the quotient thus obtained the sign +, if different, the sign —.

Thus $14ab \div 2a = \frac{14ab}{2a} = 7b$, $-12a \div 10c = -\frac{12a}{10c} = -\frac{6a}{5c}$, $-a \div -2c = +\frac{a}{2c}$.

The rule for the sign of the quotient is the same as that given in (19), viz. that *like signs produce* + *and unlike* −; and is clearly derived from it, for if $+a$ *multiplied* by $-b$ produces $-ab$, of course $-ab$ *divided* by $+a$ produces $-b$; and so in the other cases.

26. One power of a quantity is divided by another by subtracting the index of the latter from that of the former.

Thus $\frac{a^5}{a^3} = a^{5-3} = a^2$; for $\frac{a^5}{a^3} = \frac{a^3 . a^2}{a^3} = a^2$: so $\frac{x^3y^2z^4}{xyz} = x^2yz^3$.

27. If the dividend contain several terms, while the divisor still consists of only one, each term of the former must be separately divided by the latter.

Ex. 1. $\frac{x^3y - 3x^2y^3 + 9y^2}{3xy} = \frac{x^3y}{3xy} - \frac{3x^2y^3}{3xy} + \frac{9y^2}{3xy} = \frac{x^2}{3} - xy^2 + \frac{3y}{x}$.

Ex. 2. $\frac{a^2c^2 - 2abc^2 + 3ac^2}{-4abc^2} = -\frac{a^2c^2}{4abc^2} + \frac{2abc^2}{4abc^2} - \frac{3ac^3}{4abc^2} = -\frac{a}{4b} + \frac{1}{2} - \frac{3c}{4b}$.

28. But if the divisor be also a compound quantity, we must proceed as in common Arithmetic, viz.

(i) Place the quantities, as in Division of Arithmetic, arranging the terms of each of them, so that the different powers of some one letter, common to both of them, may follow in order of the magnitude of their indices, (it matters not whether in *ascending* or *descending* order, only the *same* order in each of them);

(ii) Divide the first term of the dividend by that of the divisor, and set the result in the quotient.

(iii) Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend;

(iv) Bring down fresh terms (as may be required) from the dividend, and repeat the whole operation.

Ex. 1.

$$\begin{array}{l} 1-x)\ 1-2x+x^2\ (1-x \\ \quad\ \ \underline{1-\ x\qquad} \\ \quad\ \ -\ x+x^2 \\ \quad\ \ \underline{-\ x+x^2} \end{array}$$

Ex. 2.

$$\begin{array}{l} 3x-4y)\ 6x^3-17x^2y+16y^3\ (2x^2-3xy-4y^2 \\ \qquad\quad \underline{6x^3-\ 8x^2y} \\ \qquad\qquad -\ 9x^2y+16y^3 \\ \qquad\qquad \underline{-\ 9x^2y+12xy^2} \\ \qquad\qquad\qquad -12xy^2+16y^3 \\ \qquad\qquad\qquad \underline{-12xy^2+16y^3} \end{array}$$

Ex. 3.

$$\begin{array}{l} a-x)\ a^3-x^3\ (a^2+ax+x^2 \\ \qquad \underline{a^3-a^2x} \\ \qquad\quad a^2x-x^3 \\ \qquad\quad \underline{a^2x-ax^2} \\ \qquad\qquad ax^2-x^3 \\ \qquad\qquad \underline{ax^2-x^3} \end{array}$$

Ex. 4.

$$\begin{array}{l} a+x)\ a^3+x^3\ (a^2-ax+x^2 \\ \qquad \underline{a^3+a^2x} \\ \qquad\quad -a^2x+x^3 \\ \qquad\quad \underline{-a^2x-ax^2} \\ \qquad\qquad ax^2+x^3 \\ \qquad\qquad \underline{ax^2+x^3} \end{array}$$

Ex. 5.

$$\begin{array}{l} a+x)\ a^2+x^2\ (a-x+\dfrac{2x^2}{a+x} \\ \qquad \underline{a^2+ax} \\ \qquad\quad -ax+\ x^2 \\ \qquad\quad \underline{-ax-\ x^2} \\ \qquad\qquad 2x^2 \end{array}$$

Ex. 6.

$$\begin{array}{l} a-x)\ a^2+x^2\ (a+x+\dfrac{2x^2}{a-x} \\ \qquad \underline{a^2-ax} \\ \qquad\quad ax+\ x^2 \\ \qquad\quad \underline{ax-\ x^2} \\ \qquad\qquad 2x^2 \end{array}$$

In each of the last two examples we have a remainder $2x^2$, which we place in the quotient, as in Arithmetic, over the divisor, in the form of a fraction, thus indicating that $2x^2$ remains still to be divided by $a+x$, $a-x$ respectively.

In this and other cases, as in common Arithmetic, this fraction could not be avoided, since a^2+x^2 is not exactly divisible by $a+x$; but the student should be cautioned, that, unless attention is paid to the *arrangement according to powers*, alluded to above (28, i), and that, not only with the dividend and divisor at starting, but also throughout the sum, care being taken in all the remainders to preserve the order of the indices of the principal letter, or *letter of reference*, as it is called, there will *always* be a fractional term of this kind, instead of a clear and complete quotient.

Ex. 12.

1. Divide ab^2c^3 by abc, $-165x^2y^5$ by $-33xy^3$, $-70abx^3y$ by $2ax^2y$.
2. Divide $6x^2y-4x^2z+6xyz$ by $2x$, $5a^3b^3-35a^2b^2c^2+20abc^4$ by $-5ab$, and $a^3x^3y-3a^2bx^2y+3ab^2xy^2-b^3xy^3$ by $abxy$.

3. Divide $2m^2n^2 - 3mn^3 + 4m^3n - n^4$ by $-3mn^3$, and $-3a^4b + 5a^3b^2 - 6a^2b^3 - ab^4 + 4b^5$ by $-2a^2b^3$.

4. Divide $x^2 + 6x + 5$ by $x + 1$, and $m^3 - 6m^2 + 11m - 6$ by $m - 2$.

5. Divide $6a^2 - 16ab + 8b^2$ by $2a - 4b$, and $6x^2 + 13xy + 6y^2$ by $2x + 3y$.

6. Divide $6a^2b^2 - ab^3 - 12b^4$ by $3ab + 4b^2$.

7. Divide a^4+4b^4 by $a^2+2ab+2b^2$, and $4x^4y^4+1$ by $2x^2y^2-2xy+1$.

8. Divide $x^6-2x^5y+2x^4y^2-4x^3y^3+8x^2y^4+16xy^5-32y^6$ by x^2-2y^2.

9. Divide $1+6x^5+5x^6$ by $1+2x+x^2$, and a^6-6a+5 by a^2-2a+1.

10. Divide $x^4-4xy^3+3y^4$ by $x^2-2xy+y^2$, and m^4+4m+3 by m^2+2m+1.

11. Divide $a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$ by $a^2-2ab-3b^2$.

12. Divide x^6-2x^3+1 by x^2-2x+1, and $a^6+2a^3b^3+b^6$ by $a^2+2ab+b^2$.

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29. In some of the following Examples, the div^r and div^d are not properly arranged according to powers: the student must attend to this *before* and *in the course of* division. In Ex. 1, for instance, where a is taken as the letter of reference, and its powers arranged in descending order, there is found in the first rem^r the terms $-a^2b$, $-a^2c$. These terms must be set *first*, but since both involve a^2, there is nothing as far as a is concerned to shew which is to be set *first of the two*. In such cases we take another letter, as b, to be, as it were, next in authority to a, and so, (arranging in descending powers of b,) we prefer $-a^2b$ to $-a^2c$.

Ex. 1. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

$$
\begin{array}{l}
a + b + c)\ a^3 - 3abc + b^3 + c^3\ (a^2 - ab - ac + b^2 - bc + c^2 \\
\qquad \underline{a^3 + a^2b + a^2c} \\
\qquad\quad - a^2b - a^2c - 3abc \\
\qquad\quad \underline{- a^2b - ab^2 - abc} \\
\qquad\qquad - a^2c + ab^2 - 2abc \\
\qquad\qquad \underline{- a^2c - abc - ac^2} \\
\qquad\qquad\quad + ab^2 - abc + ac^2 + b^3 \\
\qquad\qquad\quad \underline{+ ab^2 + b^3 + b^2c} \\
\qquad\qquad\qquad - abc + ac^2 - b^2c \\
\qquad\qquad\qquad \underline{- abc - b^2c - bc^2} \\
\qquad\qquad\qquad\quad + ac^2 + bc^2 + c^3 \\
\qquad\qquad\qquad\quad \underline{+ ac^2 + bc^2 + c^3}
\end{array}
$$

The above is the most *easy* method in such a case; but the following, in which the coeff^s of the different powers of a are collected in brackets, is the most neat and compendious.

Ex. 2. $a + (b + c)]\ a^3 - 3abc + (b^3 + c^3)\ [a^2 - (b + c)\,a + (b^2 - bc + c^2)$

$$\begin{array}{l} a^3 + (b+c)\,a^2 \\ \hline -(b+c)\,a^2 - 3bca \\ -(b+c)\,a^2 - (b^2+2bc+c^2)\,a \\ \hline +(b^2-bc+c^2)\,a + (b^3+c^3) \\ +(b^2-bc+c^2)\,a + (b^3+c^3) \\ \hline \end{array}$$

Ex. 13.

1. Divide $x^3 - (a + p)\,x^2 + (q + ap)\,x - aq$ by $x - a$.
2. Divide $maz^3 + (mb - na)\,z^2 - (mc + nb)\,z + nc$ by $mz - n$.
3. Divide $y^5 - my^4 + ny^3 - ny^2 + my - 1$ by $y - 1$.
4. Divide $a^2 - b^2 - c^2 + d^2 - 2ad + 2bc$ by $a - b + c - d$.
5. Divide $a^2 + ab + 2ac - 2b^2 + 7bc - 3c^2$ by $a - b + 3c$.
6. Divide $a^3 - b^3 + c^3 + 3abc$ by $a - b + c$, and $a^3 - b^3 - c^3 - 3abc$ by $a - b - c$.
7. Divide $1 + x^3 - 8y^3 + 6xy$ by $1 + x - 2y$, and $1 - x^3 + 8y^3 + 6xy$ by $1 - x + 2y$.
8. Divide $x^3 - 8y^3 - 27z^3 - 18xyz$ by $x - 2y - 3z$.
9. Divide $x^4 + y^4 - z^4 + 2x^2y^2 - 2z^2 - 1$ by $x^2 + y^2 - z^2 - 1$.
10. Divide a by $1 + x$, and $1 + 2x$ by $1 - 3x$, each to 4 terms in the quotient.
11. Divide 1 by $1 - 2x + x^2$, and $1 - ax$ by $1 + bx$, each to 4 terms.
12. Find the rem^r, when $x^3 - px^2 + qx - r$ is divided by $x - a$.

30. We have seen above (28 Ex. 3 and 4) that $a^3 - x^3$ is exactly divisible by $a - x$, and $a^3 + x^3$ by $a + x$, and that in the quotient in each case, the powers of a *decrease* continually, while those of x *increase*. The following general facts should be well noticed, as they will enable us to write down at once the quotient, when similar cases occur, as they often do, in practice. It will be seen that the index n is here used to denote generally *any* index, as the case may be: the quantity a^n is called the n^{th} *power of* a, and read *a to the n^{th}*.

If the index be *odd*, $a^n + x^n$ (like $a + x$) is div. by $a + x$,
$a^n - x^n$ (like $a - x$) . . . . by $a - x$;
if the index be *even*, $a^n + x^n$ (like $a^2 + x^2$) . . . . by *neither*,
$a^n - x^n$ (like $a^2 - x^2$) . . . . by *both*.

The student will best remember these, by thinking, in each case, of the *simplest* form of the same kind.

Thus, for $a^4 + x^4$ (index *even*, sign +) let him think of $a^2 + x^2$; this, he knows, is div. by *neither;* then a^4+x^4 is div. by neither: again, for $a^5 - x^5$ (index *odd*, sign −) let him think of $a - x$; this is, of course, divisible by $a - x$; then $a^5 - x^5$ is so divisible: for $a^6 - x^6$, let him think of $a^2 - x^2$; this is divisible by *both* $a + x$ and $a - x$: then $a^6 - x^6$ is divisible by both.

Now, in every case, the quotient will consist (as may be seen by actual division) of terms, in which the powers of a decrease, and of x increase continually: but when the divʳ is $a-x$, these terms are all +; when it is $a+x$, they are alternately + and −.

Thus, $\dfrac{a^4 - x^4}{a - x} = a^3 + a^2x + ax^2 + x^3$, $\dfrac{a^4 - x^4}{a + x} = a^3 - a^2x + ax^2 - x^3$,

$$\frac{a^5 + x^5}{a + x} = a^4 - a^3x + a^2x^2 - ax^3 + x^4.$$

The above results may now be applied to many similar cases.

Ex. 1. $\dfrac{8a^3x^3-1}{2ax-1} = 4a^2x^2+2ax+1.$ Ex. 2. $\dfrac{x^3+27y^3}{x+3y}=x^2-3xy+9y^2.$

Ex. **14.**

Write down the quotients

1. Of $a^2 - x^2$ by $a + x$, $a^5 - x^5$ by $a - x$, and $a^6 - x^6$ by $a - x$.
2. Of $9x^2 - 1$ by $3x - 1$, $25x^2 - 1$ by $5x + 1$, and $4x^2 - 9$ by $2x+3$.
3. Of $9m^2n^2 - 25$ by $3mn + 5$, and $16m^4 - n^4$ by $4m^2 + n^2$.
4. Of $1+8x^3$ by $1+2x$, $27x^3 - 1$ by $3x - 1$, and $1-16x^4$ by $1+2x$.
5. Of x^4-81y^4 by $x-3y$, a^5+32b^5 by $a+2b$, and $x^{18}-y^{12}$ by x^3+y^2.
6. Of $\frac{1}{8}a^3 + b^3$ by $\frac{1}{2}a + b$, and $x^4y^4 - z^4$ by $xy + z$.
7. Of $(a + b)^2 - c^2$ by $a + b - c$, and $a^2 - (b - c)^2$ by $a - b + c$.
8. Of $(x + y)^3 + z^3$ by $x + y + z$, and $x^3 - (y - z)^3$ by $x - y + z$.

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31. The above results and those of (23) may also be applied to resolve algebraical quantities into their elementary factors, a process which is often required.

Ex. 1. $4x^2 - y^2 = (2x + y)(2x - y)$: $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.

Ex. 2. $(2a-b)^2-(a-2b)^2=(2a-b+a-2b)(2a-b-a+2b)=3(a-b)(a+b)$.

Ex. 3. $x^6-a^6=(x^3+a^3)(x^3-a^3)=(x+a)(x^2-ax+x^2)(x-a)(x^2+ax+x^2)$.

Ex. 4. $(a^3 - x^3)^2 = \{(a - x)(a^2 + ax + x^2)\}^2 = (a - x)^2 \times (a^3+ax+x^2)^2$.

Ex. **15.**

Resolve into elementary factors

1. $1-4x^2$, a^2-9x^2, $9m^2-4n^2$, $25a^2x^2-4x^2$, $16x^4y^2-25x^2y^4$.
2. x^3+y^3, x^3-y^3, $1+x^3y^3$, x^4-1, $a^2xy^3-x^5y$, $2a^3b^2c-8ab^2c^3$.
3. $25x^5-a^2x^3$, $a^6-9a^4b^6$, $8x^3-27$, a^3-8b^3, $a^3x^2y+27x^2y^4$.
4. x^5+32, $a^3x^3+27x^6$, $8x^9+y^6$, $a^4b^{12}-c^8$, $a^3bc+2a^2bc^2+abc^3$.
5. $81x^4-1$, x^6-64, $x^4-2bx^3+b^2x^2$, $x^6-2a^2x^4+a^4x^2$.
6. $(3x-2)^2-(x-3)^2$, $(a+b)^2-4b^2$, $(4x+3y)^2-(3x+4y)^2$.
7. $(x^2+y^2)^2-4x^2y^2$, $c^2-(a-b)^2$, $(2a+b)^2-(2a-b)^2$.
8. $x^3+y^3+3xy(x+y)$, $m^3-n^3-m(m^2-n^2)+n(m-n)^2$, $a^2-ab+2(b^2-ab)+3(a^2-b^2)-4(a-b)^2$.
9. $5(x^2-y^2)+3(x+y)^2$, $3(x^2-y^2)-5(x-y)^2$, $(x+y)^2+2(x^2+xy)-3(x^2-y^2)$.
10. $2(a^3+a^2b+ab^2)-(a^3-b^3)$, $a^3-b^3-3ab(a-b)$, $a^4-b^4+(a^2-b^2)^2-2a^4+2a^2b^2$.

32. So too we may often apply (23 iii) to resolve a *trinomial* into factors.

Ex. 1. $x^2+7x+12=(x+3)(x+4)$. Ex. 2. $x^2-9x+14=(x-2)(x-7)$.
Ex. 3. $x^2-5x-14=(x-7)(x+2)$. Ex. 4. $6x^2+x-12=(3x-4)(2x+3)$.

The student may notice that, if the last term of the given trinomial be *positive*, (Ex. 1, 2), then the last terms of the two factors will have the same sign as the middle term of the trinomial; but if *negative*, (Ex. 3, 4), they will have one the sign +, the other −.

In Ex. 4, it is clear that the first terms of the two factors might be $6x$ and x, or $3x$ and $2x$, since the product of either of these pairs is $6x^2$; and so the last two terms might be 12 and 1, 6 and 2, or 4 and 3: it is easily seen on trial which are to be taken, that is, which serve also to produce the *middle* term of the trinomial.

Ex. **16.**

Resolve into elementary factors

1. x^2+6x+5, $x^2+9x+20$, x^2-5x+6, $x^2-8x+15$, x^2+8x+7, $x^2-10x+9$.
2. x^2+x-6, x^2-x-6, x^2-2x-3, $x^2+2x-15$, x^2+7x-8, x^2-8x-9.
3. $4x^2+8x+3$, $4x^2+13x+3$, $4x^2+11x-3$, $4x^2-4x-3$, $3x^2+4x-4$, $6x^2+5x-4$.
4. $12x^2-5x-2$, $12x^2-14x+2$, $12x^2-x-1$, x^2+x-12, $3x^2-2x-5$.
5. $a^2x^2-3a^3x+2a^4$, $a^3-a^2x+6ax^2$, $3a^3b+a^2b^2-2ab^3$, $12a^4+a^2x^2-x^4$.
6. $2x^3y+5x^2y^2+2xy^3$, $9x^2y^2-3xy^3-6y^4$, $6a^4x^2+a^2x-a^2$, $6b^2x^2-7bx^3-3x^4$.

CHAPTER III.

SIMPLE EQUATIONS.

33. WHEN two algebraical quantities are connected by the sign $=$, the whole expression is called, according to circumstances, an *identity* or an *equation*.

An *identity* is merely the statement of the equivalence of two different forms of the same quantity, and is true for *any* values of any of the letters involved in it.

Thus it is *always* true, whatever be the values of x and y, that $(x+y)(x-y)=x^2-y^2$, or that $(x \pm y)^2 = x^2 \pm 2xy + y^2$: and so also it is always true that $\frac{1}{2}(x+y)+\frac{1}{2}(x-y)=\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}x-\frac{1}{2}y=x$, and, in like manner, that $\frac{1}{2}(x+y)-\frac{1}{2}(x-y)=\frac{1}{2}x+\frac{1}{2}y-\frac{1}{2}x+\frac{1}{2}y=y$: each of these expressions is therefore an identity.

And in this way we may see one of the principal advantages of Algebra, viz., that it enables us to prove once for all, and express by means of letters as *general* statements, results which by mere Arithmetic we could only shew to be true upon actual trial in each instance.

Of this we have seen examples in the three formulæ of (23); and so also the two last above given express the general facts, that the greater (x) of any two quantities is equal to the *sum*, and the lesser (y) is equal to the *difference*, of their semi-sum and semi-difference.

34. An *equation*, however, is the statement of the equality of two *different* algebraical quantities; in which case the equality does not exist for *any*, but only for some particular values of one or more of the letters contained in it.

Thus the equation, $x-3=4$, will be found true only when we give x the value 7, and $x^2=3x-2$ only when we give x the value 1 or 2.

We are about to explain the method of finding these values which satisfy the simpler kinds of equations.

35. The last letters of the alphabet, x, y, z, &c., are usually employed to denote those quantities, to which particular values are to be given in order to satisfy the equation, and are said to be the *unknown quantities.*

An equation is said to be *satisfied* by any value of the unknown quantity which makes the values of the *two* sides of the equation the *same.*

This includes the case when all terms of an equation lie on one side and 0 on the other, as in $x^2 - 3x + 2 = 0$, which is satisfied by 1 or 2, either of which, being put for x, makes the first side $= 0$.

Those values of the unknown quantities, by which the equation is satisfied, are called its *roots.*

Thus 1 and 2 are the roots of the equation $x^2 - 3x + 2 = 0$, 7 is the root of $x - 3 = 4$, 1, 2, and 3 are the roots of $x^3 - 6 = 6^2 - 11x$, &c.

36. An equation of one unknown quantity is said to be of as many *dimensions* as is denoted by the index of the highest power of the unknown involved in it.

Thus $x - 3 = 4$ is an equation of one dimension, or a *simple* equation; $x^2 = 3x - 2$ is of two dimensions, or a *quadratic* equation; $x^3 - 6 = 6x^2$ is of three dimensions, or a *cubic* equation; $x^4 - 4x = 13$ is of four dimensions, or a *biquadratic* equation; &c., &c.

It may be noticed, in passing, that it can be proved that every equation of one unknown quantity has as many roots as it has dimensions, and no more.

37. *Every term of each side of an equation may be multiplied or divided by the same quantity, without destroying the equality expressed by it.*

Thus, if $3x + \frac{5}{4}x = 34$, multiplying every term by 4, we have

$$12x + 5x = 136, \text{ or } 17x = 136;$$

therefore also, dividing each term by 17, $x = \frac{136}{17} = 8$.

Again, if $12x + 6x = 144$, dividing every term by 6,

$$2x + x = 24, \text{ or } 3x = 24;$$

hence also, dividing each term by 3, $x = 8$.
We find, therefore, that 8 is the root of each of these two equations.

38. Hence an equation may be *cleared of fractions*, by multiplying every term by any common multiple of all the den^rs^. If the L. C. M. be employed, the equation will be expressed in most simple terms.

Thus, if $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x = 13$, multiplying every term by 12, which is the L. C. M. of 2, 3, 4, we have

$$\tfrac{12}{2}x + \tfrac{12}{3}x + \tfrac{12}{4}x = 156, \text{ or } 6x + 4x + 3x = 156;$$

$$\text{hence } 13x = 156, \text{ and } x = \tfrac{156}{13} = 12.$$

39. *A quantity may be transferred from one side of an equation to the other by changing its sign, without destroying the equality expressed by it.*

Thus if $x - a = y + b$, adding a to each side of the equation (which, of course, will not destroy the equality) we have $x = y + b + a$, and, subtracting b from each side, we have $x - b = y + a$; where we see that the $-a$ has been transferred to the other side with its sign changed to $+$, and so also the $+b$, with its sign changed to $-$.

Hence if the signs of *all* the terms of both sides of an equation be changed, the equality expressed by it will not be destroyed.

Simple Equations of one Unknown Quantity.

40. To solve a simple equation of one unknown.

(1) Clear it, when necessary, of fractions (38);

(2) Collect all the terms involving the unknown quantity on one side of the equation, and the known quantities on the other, transposing them, when necessary, with change of sign (39);

(3) Add together the terms of each side, and divide the sum of the known quantities by the sum of the coefficients of the unknown quantity; and thus the root required will be found.

Ex. 1. $4x + 2 = 3x + 4$.

There being no fractions here, we have only to collect the terms; $\therefore 4x - 3x = 4 - 2$, or $x = 2$, the root of the equation.

Ex. 2. $4x + 5 = 10x - 16$.

Here $10x - 4x = 5 + 16$; $\therefore 6x = 21$, and $x = \frac{21}{6} = 3\frac{3}{6} = 3\frac{1}{2}$.

Ex. 3. $5(x + 1) - 2 = 3(x - 5)$.

Here, removing the brackets, $5x + 5 - 2 = 3x - 15$;

$\therefore$ collecting terms, $5x - 3x = -15 - 5 + 2$, or $2x = -18$, and $\therefore x = -9$.

Ex. 4. $bx + 2x - a = 3x + 2c$.

Here $bx + 2x - 3x = (b - 1)x = a + 2c$; $\therefore x = \dfrac{a + 2c}{b - 1}$.

Ex. 17.

1. $4x - 2 = 3x + 3$.
2. $3x + 7 = 9x - 5$.
3. $4x + 9 = 8x - 3$.
4. $3 + 2x = 7 - 5x$.
5. $x = 7 + 15x$.
6. $mx + a = nx + d$.
7. $3(x - 2) + 4 = 4(3 - x)$.
8. $5 - 3(4 - x) + 4(3 - 2x) = 0$.
9. $13x - 21(x - 3) = 10 - 21(3 - x)$.
10. $5(a + x) - 2x = 3(a - 5x)$.
11. $3(x - 3) - 2(x - 2) + x - 1 = x + 3 + 2(x + 2) + 3(x + 1)$.
12. $2x - 1 - 2(3x - 2) + 3(4x - 3) - 4(5x - 4) = 0$.
13. $(2 + x)(a - 3) = -4 - 2ax$.
14. $(m + n)(m - x) = m(n - x)$.

Ex. 5. $\frac{1}{2}x - \frac{2}{3}x + \frac{3}{4}x = 11 + \frac{1}{8}x$.

Here we first clear the equation of fractions, by multiplying every term by 24, the L. C. M. of the den^rs, and (observing that in the first fraction $\frac{24}{2} = 12$, in the second, $\frac{24}{3} = 8$, and so in the others) thus we get $12x - 8 \times 2x + 6 \times 3x = 264 + 3x$, or $12x - 16x + 18x = 264 + 3x$ collecting terms,

$12x - 16x + 18x - 3x = 264$; $\therefore 11x = 264$, and $x = \frac{264}{11} = 24$.

Ex. 6. $\frac{1}{2}(x + 1) + \frac{1}{3}(x + 2) = 16 - \frac{1}{4}(x + 3)$.

Multiplying by 12, we have

$6(x + 1) + 4(x + 2) = 192 - 3(x + 3)$, or $6x + 6 + 4x + 8 = 192 - 3x - 9$; collecting

$6x + 4x + 3x = 192 - 9 - 6 - 8$; $\therefore 13x = 169$, and $x = \frac{169}{13} = 13$.

Ex. 7.

$\frac{1}{2}(3x^2+x)-\frac{1}{3}(2x^2+x)+\frac{1}{4}(x^2+x)-2\frac{3}{20}=x^2+\frac{2}{15}+\frac{1}{6}(x^2+x)-\frac{1}{12}(x^2+5x)$.

Expressing the mixed number $2\frac{3}{20}$ as an improper fraction $\frac{43}{20}$, we then multiply by 60, the L. C. M. of the den^rs; and, *observing the remark at the end of* (16), we thus obtain

$90x^2+30x-40x^2-20x-15x^2-15x-129=60x^2+8+10x^2+10x-5x^2-25x$;

collecting, we find that the terms involving x^2 destroy one another (otherwise the equation would be a quadratic), and we have the result

$30x-20x+15x-10x+25x=129+8$; $\therefore 40x=137$, and $x=3\frac{17}{40}$.

Ex. 18.

1. $\frac{1}{2}x+\frac{1}{3}x=x-7$. 2. $\frac{1}{2}x-\frac{1}{3}x=\frac{1}{4}x-1$. 3. $\frac{1}{2}x-\frac{1}{3}x+\frac{1}{4}x=2-\frac{1}{6}x+\frac{5}{12}x$.
4. $\frac{2}{5}x+\frac{1}{3}(x-2)=2x-7$. 5. $\frac{2}{7}x+\frac{1}{6}(x-1)=x-4$.
6. $\frac{1}{2}(9-2x)=\frac{3}{2}-\frac{1}{10}(7x-18)$. 7. $x+\frac{1}{3}(14-x)=\frac{1}{2}(21-x)$.
8. $2x-\frac{1}{3}=\frac{2}{5}(3-2x)+\frac{1}{2}x$. 9. $\frac{1}{7}(2x+7)-\frac{1}{11}(9x-8)=\frac{1}{2}(x-11)$.
10. $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=0$.
11. $\frac{1}{2}(3x-1)-\frac{2}{3}(x-1)=\frac{1}{4}(x-3)-\frac{1}{6}(x-5)+5\frac{1}{3}$.
12. $\frac{1}{6}x-1\frac{2}{3}=8\frac{3}{5}+2(\frac{3}{5}x-1)-\frac{1}{3}(x+8)$.

In some of the common examples the common multiple of all the den^rs is too large to be conveniently employed. In such a case, we may see whether two or three of the den^rs have a simple common multiple, and get rid of their fractions first, observing to collect terms, and simplify as much as possible, after each step.

Ex. 8. $\frac{1}{11}(2x+3)-\frac{1}{3}(x-12)+\frac{1}{4}(3x+1)=5\frac{1}{3}+\frac{1}{12}(4x+3)$.

Here the L. C. M. of all the den^rs would be 132: but as 12 will include three of them, multiplying by it, (having first changed $5\frac{1}{3}$ to $\frac{16}{3}$), we get

$$\frac{12}{11}(2x+3)-4(x-12)+3(3x+1)=64+4x+3;$$

$$\therefore \frac{12}{11}(2x+3)-4x+48+9x+3=64+4x+3;$$

hence, collecting terms and simplifying, we have

$\frac{12}{11}(2x+3)-4x+9x-4x=64+3-48-3$, or $\frac{12}{11}(2x+3)+x=16$;

$$\therefore 12(2x+3)+11x=176, \text{ or } 24x+11x=176-36;$$

$$\therefore 35x=140, \text{ and } x=\tfrac{140}{35}=4.$$

Ex. **19**.

1. $\frac{1}{12}(2x-3) - \frac{1}{5}(3x-2) = \frac{1}{8}(4x-3) - 3\frac{5}{24}$.
2. $\frac{5}{7}(x-9) + \frac{7}{9}(x-5) = \frac{9}{5}(x-7) + 1\frac{2}{3}$.
3. $\frac{1}{15}(2x-1) - \frac{1}{16}(3x-2) = \frac{1}{18}(x-12) - \frac{1}{24}(x+12)$.
4. $\frac{1}{8}(7x+20) - \frac{3}{16}(3x+4) = \frac{1}{10}(3x+1) - \frac{1}{20}(29-8x)$.
5. $\frac{3}{4}(a-2x) - \frac{2}{3}(2a-x) + \frac{1}{8}(x-a) = \frac{15}{32}(a+x)$.
6. $\frac{1}{11}(9x-10) - \frac{1}{13}(2x-7) = \frac{2}{3}x - \frac{1}{33}(5+x)$.
7. $\frac{7}{15}(4x-1) - \frac{5}{24}(2x+1) = 5\frac{1}{4} - \frac{1}{32}x$.
8. $\frac{2}{3}\{a-(b-x)\} - \frac{3}{4}\{x-(b-a)\} - \frac{4}{5}\{b-(a+x)\} = \frac{5}{6}\{x+a-b\}$.
9. $\frac{1}{7}(4x-21) + 7\frac{5}{6} + \frac{7}{3}(x-4) = x + 3\frac{3}{4} - \frac{1}{8}(9-7x) + \frac{1}{12}$.
10. $\frac{1}{5}(x-a) - \frac{1}{24}\{m-(a-x)\} = \frac{3}{16}(m+x) - \frac{1}{80}(15a+16m)$.
11. $\frac{1}{27}(2x+7) - \frac{1}{15}(2x-7) = 1\frac{5}{6} - \frac{1}{20}(3x+4)$.
12. $\frac{1}{3}\{x-(2a-3c)\} - \frac{5}{72}\{7a-5(x-2c)\} = \frac{1}{48}\{8(a+10c)-(2c-x)\}$.

Problems producing Simple Equations.

41. We shall now see the practical application of the above in the solution of many entertaining Arithmetical questions. In treating these, however, after having observed the methods used in the following examples, the student must be left very much to his own ingenuity, as no *general* rule can be stated for their solution. The only advice that can be given is to read over carefully and consider well the meaning of the question proposed; then it will always appear that some quantity, at present unknown, is required to be found from the *data* furnished by it: put x to represent this quantity, and now set down in algebraical language the statements made in the question, using x whenever this unknown quantity is wanted in it. We shall thus (in the problems we are now considering) arrive at a simple equation, by means of which the value of x may be found.

Ex. 1. What number is that to which if 8 be added, one-fourth of the sum is equal to 29;

Let x represent the number required;

adding 8 to it, we have $x + 8$, one-fourth of this is $\frac{1}{4}(x + 8)$, and this is equal to 29;

we have, therefore, the equation $\frac{1}{4}(x + 8) = 29$, whence $x = 108$.

Ex. 2. What number is that, the double of which exceeds its half by 6?

Let $x =$ the number;

then the double of x is $2x$, the half of x is $\frac{1}{2}x$;

hence $2x - \frac{1}{2}x = 6$, whence $x = 4$.

Ex. 3. A cask, which held 270 gallons, was filled with a mixture of brandy, wine, and water. There were 30 gallons of wine in it more than of brandy, and 30 of water more than there were of wine and brandy together. How many were there of each?

Let $x =$ no. of gals. of brandy;

$\therefore x + 30 =$ . . . . . . wine,

and $2x + 30 =$ . . . . . . wine and brandy together;

$\therefore 2x + 30 + 30$ or $2x + 60 =$ gals. of water;

but the whole number of gallons was 270;

$$\therefore x + (x + 30) + (2x + 60) = 270,$$

whence $x = 45$, the no. of gals. of brandy,

$x + 30 = 75$, ... wine,

$2x + 60 = 150$, ... water.

Ex. 4. A sum of £50 is to be divided among A, B, and C, so that A may have 13 guineas more than B, and C £5 more than A: determine their shares.

Let $x = B$'s share in *shillings:*

$\therefore x + 273 = A$'s, and $(x + 273) + 100$ or $x + 373 = C$'s;

$\therefore$, since £50 = 1000*s*, $(x + 273) + x + (x + 373) = 3x + 646 = 1000$;

$\therefore 3x = 354$, and $x = 118$, $x + 273 = 391$, $x + 373 = 491$,

and the shares are 391*s*, 118*s*, 491*s*, or £19 11*s*, £5 18*s*, £24 11*s*, respectively.

Ex. 5. A, B, C divide among themselves 620 cartridges, A taking 4 to B's 3, and 6 to C's 5: how many did each take?

Let $x = A$'s share; then $\frac{3}{4}x = B$'s, $\frac{5}{6}x = C$'s:

$$\therefore x + \tfrac{3}{4}x + \tfrac{5}{6}x = 620, \text{ whence } x = 240,\ \tfrac{3}{4}x = 180,\ \tfrac{5}{6}x = 200.$$

We might have avoided fractions by assuming $12x$ for A's share, when we should have had $9x = B$'s, and $10x = C$'s;

$$\therefore 12x + 9x + 10x = 620, \text{ whence } x = 20;$$

and the shares are 240, 180, 200, as before.

Ex. 20.

1. What number is that which exceeds its sixth part by 10?

2. What number is that, to which if 7 be added, twice the sum will be equal to 32?

3. Find a number such that its half, third, and fourth parts shall be together greater than its fifth part by 106.

4. A bookseller sold 10 books at a certain price, and afterwards 15 more at the same rate, and at the latter time received 35*s.* more than at the former: what was the price per book?

5. What two nos are those, whose sum is 48 and difference 22?

6. At an election where 979 votes were given, the successful candidate had a majority of 47; what were the numbers for each?

7. *A* spent 2*s* 6*d* in oranges, and says, that 3 of them cost as much under 1*s*, as 9 of them cost over 1*s*: how many did he buy?

8. The sum of the ages of two brothers is 49, and one of them is 13 years older than the other: find their ages.

9. Find a number such that if increased by 10, it will become five times as great as the third part of the original number.

10. Divide 150 into two parts, so that one of them shall be two-thirds of the other.

11. A post is a fourth of its length in the mud, a third of its length in the water, and 10 feet above the water: what is its length?

12. There is a number such that, if 8 be added to its double, the sum will be five times its half. Find it.

13. Divide 87 into three parts, such that the first may exceed the second by 7, and the third by 17.

14. Find a number such that, if 10 be taken from its double, and 20 from the double of the remainder, there may be 40 left.

15. A market-woman being asked how many eggs she had, replied, If I had as many more, half as many more, and one egg and a half, I should have 104 eggs: how many had she?

16. *A* and *B* began to play with equal sums; *A* won 30*s*, and then 7 times *A*'s money was equal to 13 times *B*'s: what had each at first?

17. *A* is twice as old as *B*; twenty-two years ago he was three times as old. Required *A*'s present age.

18. *A* and *B* play together for a stake of 5*s*; if *A* win, he will

have thrice as much as *B;* but if he lose, he will have only twice as much. What has each at first?

19. Divide £64 among three persons, so that the first may have three times as much as the second, and the third, one-third as much as the first and second together.

20. A workman is engaged for 28 days at 2*s* 6*d* a day, but instead of receiving anything, is to pay 1*s* a day, on all days upon which he is idle: he receives altogether £2 12*s* 6*d*; for how many idle days did he pay?

21. A person buys 4 horses, for the second of which he gives £12 more than for the first, for the third £6 more than for the second, and for the fourth £2 more than for the third. The sum paid for all was £230. How much did each cost?

22. A person bought 20 yards of cloth for 10 guineas, for part of which he gave 11*s* 6*d* a yard, and for the rest 7*s* 6*d* a yard. How many yards of each did he buy?

23. Two coaches start at the same time from York and London, a distance of 200 miles, travelling one at $9\frac{1}{2}$ miles an hour, the other at $9\frac{1}{4}$: where will they meet, and in what time from starting?

24. A cistern is filled in 20 min. by 3 pipes, one of which conveys 10 gallons more, and another 5 gallons less than the third *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

25. *A* starts upon a walk at the rate of 4 miles an hour, and after 15′ *B* starts at the rate of $4\frac{3}{4}$ miles an hour; when and where will he overtake *A*?

26. A garrison of 1000 men was victualled for 30 days; after 10 days it was reinforced, and then the provisions were exhausted in 5 days: find the number of men in the reinforcement?

27. *A* and *B* have together 8*s*, *A* and *C* have 10*s*, *B* and *C* have 12*s*. What have they each?

28. What was the total amount of a person's debts, who when he had paid a half, and then a third, and then a twelfth of them, had still 15 guineas to pay?

29. A father's age is 40 and his son's 8: in how many years will the father's age be triple of the son's?

30. How much tea at 4*s* 6*d* must be mixed with 50 lbs. at 6*s*, that the mixture may be sold at 5*s* 6*d*?

CHAPTER IV.

INVOLUTION AND EVOLUTION.

42. Involution is the name given to the operation by which we find the *powers* of quantities. We have already (22) had occasion to notice the square of a binomial: but all cases of Involution are merely examples of Mult^n, where the factors are all the *same.*

It should be noticed, that any *power of a power* of a quantity is obtained by multiplying together the indices of the two powers.

Thus the cube of x^2, that is $(x^2)^3 = x^6$; for it $= x^2 \times x^2 \times x^2 = x^{2+2+2}$ (20) $= x^6$: and, similarly, $(x^3)^2 = x^6 = (x^2)^3$, that is, the *square of the cube* is the same as the *cube of the square* of any quantity, &c.

So also $(a^3)^4 = a^{12} = (a^4)^3$, $(2x^3y^2)^2 = 4x^6y^4$, $(-2xy^2z^3)^3 = -8x^3y^6z^9$, $\left(\frac{-3a^2b}{c^3}\right)^4 = \frac{81a^8b^4}{c^{12}}$, $\{(a-b)^3\}^2 = (a-b)^6$, $\{(x+y)^2\}^4 = (x+y)^8$, &c.

Hence, we may shorten the operation of finding the 4th power of a quantity by squaring its square; and, similarly, to find the 6th, 8th, &c. powers, we may square the 3d, 4th, &c.

So also to find the cube, or 3rd power, we may take the product of the 1st and 2d, that is, of the quantity itself and its square; to find the 5th, we may take that of the square and cube; to find the 7th, of the cube and 4th; and so on.

Thus we shall have

$(a+b)^3 = (a+b)(a^2 \pm 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3$, by Mult^n,

$(a-b)^3 = (a-b)(a^2 - 2ab + b^2) = a^3 - 3a^2b + 3ab^2 - b^3$,

$(a \pm b)^4 = (a^2 \pm 2ab + b^2)(a^2 \pm 2ab + b^2) = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$

$(a \pm b)^5 = (a^2 \pm 2ab + b^2)(a^3 \pm 3a^2b + 3ab^2 \pm b^3)$

$= a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5$.

The above results should be remembered and applied in the following Examples. The expansions of higher powers are generally best obtained by the Binomial Theorem, which will be given hereafter.

Ex. 1. $(a+b+c)^3 = \{a + (b+c)\}^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$
$= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$.

Ex. 2. $(a-b-c)^3 = \{a-(b+c)\}^3 = a^3 - 3a^2(b+c) + 3a(b+c)^2 - (b+c)^3$
$= a^3 - 3a^2b - 3a^2c + 3ab^2 + 6abc + 3ac^2 - b^3 - 3b^2c - 3bc^2 - c^3$.

Or thus:

$(a-b-c)^3 = \{(a-b)-c\}^3 = (a-b)^3 - 3(a-b)^2c + 3(a-b)c^2 - c^3$, which, of course, when expanded, would give the same result as before.

Ex. 3. $(2x-3)^4 = (2x)^4 - 4 \,.\, 3 \,.\, (2x)^3 + 6 \,.\, 3^2 \,.\, (2x)^2 - 4 \,.\, 3^3 \,.\, (2x) + 3^4$
$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$.

Ex. **21**.

1. Find the values of $(2ab^2)^2$, $(-3a^2b^2c^4)^3$, $\left(-\frac{3ab^2}{4c^3}\right)^4$, $\left(-\frac{x^2y^3z^4}{2}\right)^5$.

Write down the expansions of

2. $(x+2)^3$. 3. $(x-2)^4$. 4. $(x+3)^5$. 5. $(1+2x)^5$.
6. $(2m-1)^3$. 7. $(3x+1)^4$. 8. $(2x-a)^4$. 9. $(3x+2a)^5$.
10. $(4a-3b)^3$. 11. $(ax-y^2)^3$. 12. $(ax+x^2)^4$.
13. $(2am-m^2)^5$. 14. $(a-b+c)^3$. 15. $(1-x+x^2)^3$.
16. $(a+bx+cx^2)^3$. 17. $(1+x+x^2)^4$. 18. $(1+x-x^2)^5$.
19. $(1-2x+x^2)^3$. 20. $(a-2b+c)^4$. 21. $(1+2x-3x^2)^5$.

---

43. The following result is worthy of notice, as it exhibits the form of the square of any Multinomial.

$$\begin{aligned}(a+b+c+d+\&c.)^2 &= a^2 + 2a(b+c+d+\&c.) + (b+c+d+\&c.)^2 \\ &= a^2 + 2ab + 2ac + 2ad + \&c. \\ &\quad + b^2 + 2b(c+d+\&c.) + (c+d+\&c.)^2 \\ &= a^2 + 2ab + 2ac + 2ad + \&c. \\ &\quad + b^2 + 2bc + 2bd + \&c. \qquad \text{(i)} \\ &\qquad + c^2 + 2cd + \&c. \\ &\qquad\quad + d^2 + \&c.\end{aligned}$$

$$= a^2 + (2a+b)b + \{2(a+b) + c\}c + \{2(a+b+c) + d\}d + \&c.,$$

$$= a^2 + (2a+b)b + (2a' + c)c + (2a'' + d)d + \&c., \qquad \text{(ii)}$$

if we write a' for $a+b$, a'' for $a+b+c$, &c.

We see from (i) that the square of any multinomial may be formed by setting down the *square of each term* and then *the product of the double of each term by the sum of all the terms that follow it.*

Another form of this result is given in (ii), to which reference will be made hereafter.

Ex. 1. $(1+2x+3x^2)^2 = 1 + 2(2x+3x^2) + 4x^2 + 4x(3x^2) + 9x^4$
$= 1 + 4x + 10x^2 + 12x^3 + 9x^4.$

Ex. 2.

$$(a+bx+cx^2+dx^3+ex^4+\&c.)^2 = a^2+2abx+2acx^2+2adx^3+2aex^4+\&c.$$
$$+\ b^2x^2+2bcx^3+2bdx^4+\&c.$$
$$+\ c^2x^4+\&c.$$
$$=a^2+2ab\,x+(2ac+b^2)\,x^2+2\,(ad+bc)\,x^3+\{2\,(ae+bd)+c^2\}\,x^4+\&c.$$

Ex. 3. $(1-2x)^6 = (1-6x+12x^2-8x^3)^2$
$= 1 - 12x + 24x^2 - 16x^3$
$+ 36x^2 - 144x^3 + 96x^4$
$+ 144x^4 - 192x^5 + 64x^6$
$= 1 - 12x + 60x^2 - 160x^3 + 240x^4 - 192x^5 + 64x^6.$

Ex. 22.

Find the expansions of

1. $(1+x+x^2)^2$. 2. $(1-x+2x^2)^2$. 3. $(3-2x+x^2)^2$.
4. $(a^2-2ab+3b^2)^2$. 5. $(2x-3y+4z)^2$. 6. $(3ax+2by+cz)^2$.
7. $(1-2ax-a^2x^2)^2$. 8. $(2a^2-a-2)^2$. 9. $(1-x+x^2-x^3)^2$.
10. $(1+x)^6$. 11. $(x^3-2x^2+3x+4)^2$. 12. $(1+2x-3x^2+4x^3)^2$.
13. $(a^3-2a^2b+2ab^2-b^3)^2$. 14. $(a-x)^8$.
15. $(1-2x+3x^2-2x^3+x^4)^2$. 16. $(a^4-2a^3x+a^2x^2-2ax^3+x^4)^2$.

---

44. Let the student notice the following remarks:

(i) Since any *even* number of like signs, whether all + or all —, will give + in mult^n, it follows that any *even* power of a quantity is the same, whether that quantity be taken positively or negatively; thus, $(+a)^2$ and $(-a)^2$ are each $=+a^2$, and $(1-x+x^2)^4$ is the same as $\{-(1-x+x^2)\}^4$, or $(-1+x-x^2)^4$

(ii) No *even* power of *any* quantity can be *negative;*

(iii) Any *odd* power will have the same sign as the quantity itself.

45. EVOLUTION is the name given to the operation by which we find the *roots* of quantities.

Since the cube *power* of $a^2 = a^6$, therefore the cube *root* of a^6 is a^2; so $\sqrt{a^4} = a^2$, $\sqrt[4]{16a^8b^4} = 2a^2b$, &c.; and so we may often extract a required root of a simple quantity, by dividing its index by that of the root.

If, however, the index of the quantity cannot be exactly divided by that of the root (as *e. g.* in the 5th root of a^2, where the 2 cannot be divided by 5,) then we cannot find the root of it; but can only *indicate* that the root *is to be* extracted, by writing down the quantity, and the sign $\sqrt{}$ before it, with the index of the root in question; as $\sqrt[5]{a^2}$, $\sqrt[3]{a^4}$. Such quantities are called *surds*, or *irrational quantities.*

46. It follows from (44), that

(i) Any *even* root of a *positive* quantity will have a double sign $\pm$;

(ii) There can be no *even* root of a negative quantity;

(iii) Any odd root of a quantity will have the same sign as the quantity itself.

Thus $\sqrt{\frac{16a^2}{9b^2}} = \pm\frac{4a}{3b}$, $\sqrt[3]{-8x^3y^6} = -2xy^2$, $\sqrt[4]{\frac{625a^4y^8}{81x^4}} = \pm\frac{5ay^2}{3x}$ &c.

Hence, when we have a surd expressing an *odd* root of a negative quantity, we may write the quantity positive under the sign of evolution, and set the negative sign outside; thus $\sqrt[5]{-x^3} = -\sqrt[5]{x^3}$, $\sqrt[3]{a^6} - \sqrt[3]{-b^2} = a^2 + \sqrt[3]{b^2}$. But this cannot be done with an *even* root of a negative quantity, such as $\sqrt{-x^3}$, which must be left as it is, and is called an *impossible* or *imaginary* quantity; the difference between surd and impossible quantities being that the former have *real* values, though we cannot exactly find them, while there *cannot* be a quantity, positive or negative, an even power of which would produce a negative quantity.

Imaginary quantities, however, are employed in some of the higher applications of Algebra; but for the present we shall leave the consideration both of these and of surd quantities.

Ex. 23.

1. Find the square roots of $4a^2b^4c^6$, $49x^4y^6z^2$, $100a^8b^{12}c^{16}$.

2. Find the square roots of $\frac{9a^2x^4y^6}{25z^2}$, $\frac{49x^2y^4}{64a^2}$, $\frac{25x^6y^{10}}{16a^2b^4}$.

3. Find $\sqrt{\frac{a^4x^4y^2}{4}}$, $\sqrt[3]{-\frac{8a^3y^6}{27x^9}}$, $\sqrt[3]{\frac{64b^6c^9}{125a^{12}}}$, $\sqrt[3]{-\frac{216a^3b^3c^{15}}{343}}$.

4. Find $\sqrt[4]{\frac{16x^4y^8}{625a^{12}}}$, $\sqrt[4]{\frac{81a^4b^8c^{12}}{256x^{16}}}$, $\sqrt[5]{\frac{32a^5b^{10}}{c^{15}}}$, $\sqrt[6]{\frac{64x^{12}y^6}{729z^{18}}}$.

46. *To find the square root of a compound quantity.*

We *know* that the square of $a+b$ is $a^2+2ab+b^2$; let us see then how from $a^2+2ab+b^2$, we might deduce its square root $a+b$.

$$\begin{array}{rl} & a^2+2ab+b^2\ (a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

Let us write down then the quantity $a^2+2ab+b^2$. Now a, the first term of the root, may be found immediately by taking the square root of its first term: set a then on the right, and then subtract a^2; we have now remaining $2ab+b^2$, and if we divide $2ab$ by $2a$, we get $+b$, the other term in the root: lastly, if we add this b to the $2a$, multiply the $2a+b$, thus formed, by b, and subtract the product, there is no remainder.

Now we may follow this plan in any other case, and if we find no remainder, the root will be exactly obtained.

Ex. 1.

$$\begin{array}{rl} & 9x^2+6xy+y^2\ (3x+y \\ & 9x^2 \\ \hline 6x+y) & 6xy+y^2 \\ & 6xy+y^2 \\ \hline \end{array}$$

Ex. 2.

$$\begin{array}{rl} & 16a^2-56ab+49b^2\ (4a-7b \\ & 16a^2 \\ \hline 8a-7b) & -56ab+49b^2 \\ & -56ab+49b^2 \\ \hline \end{array}$$

Ex. 3.

$$\begin{array}{rl} & 4a^2-4ab-b^2\ (2a-b \\ & 4a^2 \\ \hline 4a-b) & -4ab-b^2 \\ & -4ab+b^2 \\ \hline & -2b^2 \end{array}$$

Here we find a remainder $-2b^2$; we conclude, therefore, that $2a-b$ is *not* the exact root of $4a^2-4ab-b^2$, which is a surd, and can only be written $\sqrt{4a^2-4ab-b^2}$.

Ex. 24.

Find the square roots of

1. $4x^2 + 4xy + y^2$, $25a^2 - 30ab + 9b^2$, $25x^4 + 30x^3y + 9x^2y^2$.
2. $49a^2b^2 - 14a^3b + a^4$, $16x^2y^2 + 40xy^2z + 25y^2z^2$, $25a^4b^2c^2 + 10a^2bc^5 + c^8$.

47. If the root consist of more than two terms, a similar process will enable us to find it, as in the following Example; where it will be seen that the divisor at any step is obtained by doubling the quantity already found in the root, or (which amounts to the same thing and is more convenient in practice) by *doubling the last term of the preceding divisor*, and then annexing the new term of the root.

$$
\begin{array}{lll}
\text{Ex.} & & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 \;(4x^3 - 3x^2 + 2x - 1 \\
& & \underline{16x^6} \\
8x^3 - 3x^2) & & -24x^5 + 25x^4 \\
& & \underline{-24x^5 + 9x^4} \\
8x^3 - 6x^2 + 2x) & & 16x^4 - 20x^3 + 10x^2 \\
& & \underline{16x^4 - 12x^3 + 4x^2} \\
8x^3 - 6x^2 + 4x - 1) & & -8x^3 + 6x^2 - 4x + 1 \\
& & \underline{-8x^3 + 6x^2 - 4x + 1}
\end{array}
$$

48. The reason of the above method may be thus exhibited by considering the square of $a + b + c$.

$$
\begin{array}{ll}
& a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \;(a + b + c \\
& \underline{a^2} \\
2a + b) & 2ab + b^2 \\
& \underline{2ab + b^2} \\
2a' + c = 2a + 2b + c) & 2ac + 2bc + c^2 \\
& \underline{2ac + 2bc + c^2}
\end{array}
$$

Here we find, as before, the first two terms in the root, $a + b$, subtracting first a^2, and then $2ab + b^2$, that is, in all, $a^2 + 2ab + b^2$ or $(a + b)^2$. Now denote $a + b$ by a', so that $(a + b + c)^2 = (a' + c)^2 = a'^2 + 2a'c + c^2$; then we see that, at this stage of our progress, we have found a' in the root, and have subtracted a'^2, and therefore our remainder will be no other than $2a'c + c^2$. [We see, in fact, that $2ac + 2bc + c^2 = 2\ (a + b)\,c + c^2 = 2a'c + c^2$.] Just in the same way then as when, having found a and subtracted a^2, we took $2a$ for our trial-divisor in order to find b, dividing by it the first term of the *first* remainder $2ab + b^2$, so now we take $2a'$ for our trial-divisor, in order to find c, dividing by it the first term of the *second* remainder $2a'c + c^2$. We may get $2a'$ or $2a + 2b$, by merely doubling the last term of the preceding divisor; and now subtracting $2a'c + c^2$, we shall have subtracted in all $a'^2 + 2a'c + c^2$, that is, the square of $a + b + c$.

In like manner, if the root were $a + b + c + d$, we may find $a + b + c$ as before, and put it $= a''$: then $(a + b + c + d)^2 = (a'' + d)^2 = a''^2 + 2a''d + d^2$, and, as we shall have already subtracted $(a + b + c)^2$ or a''^2, the *third* remainder will be $2a''d + d^2$; and, therefore, taking $2a''$ as trial-divisor (obtained as before by doubling the last term of the preceding divisor $2a + 2b + c$), we may get d, &c.

It will be seen that the successive subtrahends in the above operation are a^2, $(2a + b)\,b$, $(2a' + c)\,c$, $(2a'' + d)\,d$, &c., and of course, the sum of them all, that is, the whole quantity subtracted, is (43 ii) $(a + b + c + d + \&c.)^2$.

49. As the 4th *power* of a quantity is the square of its square (42), so the 4th root of a quantity is the square root of its square root, and may therefore be found by the preceding rule.

Thus, if it be required to find the 4th root of $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$, the square root will be found to be $a^2 + 2ax + x^2$, and the square root of this to be $a+x$, which is therefore the 4th root of the given quantity.

50. It should be noticed as in (45) that all *even* roots have *double signs*.

Thus the square root of $a^2 + 2ab + b^2$ may be $-(a + b)$, that is, $-a - b$, as well as $a + b$: and, in fact, the first term in the root, which we found by taking the square root of a^2, might have been $-a$ as well as a, and by using this we should have obtained also $-b$.

So in 46, Ex. 1, the root may also be $-3x - y$; in 47, $-4x^3 + 3x^2 - 2x + 1$; and in all these cases we should get the two roots by giving a double sign to the first term in the root.

Ex. 25.

Find the square root

1. Of $1+4x+10x^2+12x^3+9x^4$. 2. Of $9x^4+12x^3+22x^2+12x+9$.
3. Of $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$.
4. Of $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.
5. Of $4a^4 - 12a^3 + 25a^2 - 24a + 16$.
6. Of $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4$.
7. Of $x^6 - 4x^5 + 10x^4 - 20x^3 + 2x^2 - 24x + 16$.
8. Of $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2$.
9. Of $x^6 - 4x^5y + 8x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + y^6$.
10. Of $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
11. Of $4 - 12a + 5a^2 + 14a^3 - 11a^4 - 4a^5 + 4a^6$.
12. Of $p^2 + 2pqx + (2pr + q^2)x^2 + 2(ps + qr)x^3 + (2qs + r^2)x^4 + 2rsx^5 + s^2x^6$.

Extract the 4th root

13. Of $1 - 4x + 6x^2 - 4x^3 + x^4$, and of $a^4 - 8a^3 + 24a^2 - 32a + 16$.
14. Of $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$.

Extract the 8th root

15. Of $x^8-16x^7+112x^6-448x^6+1120x^4-1792x^3+1792x^2-1024x+256$.
16. Of $a^8-8a^7b+28a^6b^2-56a^6b^3+70a^4b^4-56a^3b^5+28a^2b^6-8ab^7+b^8$.

---

51. The method of finding the square root of a numerical quantity is derived from the foregoing.

Since $1=1^2$, $100=10^2$, $10000=100^2$, &c., it follows that the square root of any number between 1 and 100 lies between 1 and 10, that is, the square root of any number having *one* or *two* figures is a number of *one* figure; so also the square root of any number between 100 and 10000, that is, having *three* or *four* figures, lies between 10 and 100, that is, is a number of *two* figures, &c. Hence, if we set a dot over every other figure of any given square number, beginning with the units-figure, the number of dots will exactly indicate the number of figures in its square root.

Ex.	$\dot{1}88\dot{6}2\dot{4}$	$(400 + 30 + 2$ (a, b, c)
	160000	..... a^2
$(2a + b)$ .... $800 + 30 = 830)$	26624	
	24900	..... $(2a + b)b$
$(2a' + c)$ .... $800 + 60 + 2 = 862)$	1724	
	1724	..... $(2a' + c)c$

Here the number of dots is three, and therefore the number of figures in the root will be three. Now the greatest square-number, contained in 18, the first *period*, as it is called), is 16, and the number evidently lies between 160000 and 250000, that is, between the squares of 400 and 500. We take therefore 400 for the first term in the root, and proceeding just as before, we obtain the whole root, $400 + 30 + 2 = 432$. The letters annexed will indicate how the different steps of the above correspond with those of the algebraical process in (48), from which it is derived.

```
     Ex. 1.
   186624 (432
   16
  83)266
     249
  862)1724
      1724

     Ex. 2.
    77841 (279
    4
  47)378
     329
  549)4941
      4941

     Ex. 3.
   10291264 (3208
   9
  62)129
     124
  6408)51264
       51264
```

The cyphers are usually omitted in practice, and it will be seen that we need only, at any step, take down the next period, instead of the whole remainder.

In Ex. 2, notice (i) that the second remainder 49 is greater than the divisor 47; this may sometimes happen, but no difficulty can arise from it, as it would be found that if instead of 7 we took 8 for the second figure, the subtrahend would be 384, which is too large: And (ii), that the last figure 7 of the first divisor, being doubled in order to make the second divisor, and thus becoming 14, causes 1 to be added to the preceding figure, 4, which now becomes 5. In fact the first divisor is $400 + 70$, which, when its second term is doubled, becomes $400 + 140$ or 540.

In Ex. 3, we have an instance of a cypher occurring in the root.

52. If the root have any number of decimal places, it is plain (by the rule for the mult^n of decimals) that the square will have *twice* as many, and therefore the number of *decimal* places in every square decimal will be necessarily *even*, and the number of decimal places in the root will be half that number. Hence, if the given square number be a decimal, and therefore one of an *even* number of places, if we set, as before, the dot upon the *units-figure*, and then over every other

figure on *both* sides of it, the number of dots to the left, will still indicate the number of *integral* figures in the root, and the number of dots to the right the number of *decimal* places.

Thus 10.291264 would be dotted $1\dot{0}.2\dot{9}1\dot{2}6\dot{4}$, the dot being first placed on the units-figure 0; and the root will have one integral and three decimal places, that is, would be (Ex. 3 above) 3.208.

If, however, the given number be a decimal of an *odd* number of places, or if there be a rem^r in any case, then there is no exact square root, but we may approximate to it as far as we please by dotting as before, (*remembering to set the dot first upon the units-figure*,) and then annexing cyphers (which by the nature of decimals will not alter the value of the number itself) and taking them down as they are wanted, until we have got as many decimal places in the root as we desire.

Ex. Find the square roots of 2 and 259.351, to three decimal places.

Ex. 1. $\dot{2}$ (1.414 &c.

```
        1
      ----
  24)100
      96
      ---
  281)400
      281
      ----
 2824)11900
      11296
      -----
```

Ex. 2. $\dot{2}5\dot{9}.\dot{3}5\dot{1}0$ (16.104 &c.

```
         1
       ----
    26)159
       156
       ---
    321)335
        321
        ---
 32204)141000
       128816
       ------
```

Ex. **26.**

Find the square roots

1. Of 177241, 120409, 4816.36, 543169, 1094116, 18671041.
2. Of 4334724, 437.6464, 1022121, 408.8484, 16803.9369.
3. Of 14356521, 5742.6084, 229.704336, 74684164, 4888521.
4. Of 17.338896, 69355584, 6595651.24, 129208689, 975312900.
5. Of 16.353936, 65415744, 25553025, 43996689, 229977225.
6. Extract to five figures the square roots of 2.5, 2000, .3, .03, 111, .00111, .004, .005.

53. *To find the cube root of a compound quantity.*

Let us consider the quantity $a^3 + 3a^2b + 3ab^2 + b^3$, which we know to be the cube of $a + b$, and see how we may get the root from it.

$$\begin{array}{l} a^3+3a^2b+3ab^2+b^3\ (a+b \\ a^3 \\ 3a^2)\ 3a^2b+3ab^2+b^3 \\ \qquad\ \ \underline{3a^2b+3ab^2+b^3} \end{array}$$

We may get a, as before, by merely taking the cube root of the first term a^3; then, subtracting a^3, we have a remainder $3a^2b+3ab^2+b^3$: by dividing the first term of this remainder by $3a^2$, we shall get b, the other term in the root, and then, if we subtract the quantity $3a^2b + 3ab^2 + b^3$, there will be no remainder.

Pursuing the same course in any other case, if there be no remainder, we conclude that we have obtained the exact cube root.

$$\begin{array}{l} 8x^3+12x^2y+6xy^2+y^3\ (2x+y \\ 8x^3 \\ 12x^2)\ 12x^2y+6xy^2+y^3 \\ \qquad\quad \underline{12x^2y+6xy^2+y^3} \end{array}$$

Here the quantity corresponding to the *trial-divisor* $3a^2$ is $3\ (2x)^2 = 12x^2$, that to $3a^2b$ is $12x^2y$, that to $3ab^2$ is $6xy^2$, and that to b^3 is y^3; so that the whole subtrahend is

$$12x^2y+6xy^2+y^3.$$

By attending however to the following hint, the subtrahend may be more easily constructed.

$$\begin{array}{lr|l} & & a^3 + 3a^2b + 3ab^2 + b^3\ (a + b \\ & & a^3 \\ 3a + b & 3a^2 & 3a^2b + 3ab^2 + b^3 \\ & (3a+b)b & \\ & 3a^2+3ab+b^2 & 3a^2b + 3ab^2 + b^3 \end{array}$$

Set down first $3a$, some little way to the left of the first remainder, and then, multiplying this by a, obtain $3a^2$ as before; by means of this trial-divisor find b, and annex it to the $3a$, so making $3a+b$; multiply this by b, and set the product $(3a+b)\ b$ or $3ab+b^2$ under the $3a^2$, and add them up, making $3a^2+3ab+b^2$; then multiplying this by b, we have $3a^2b+3ab^2+b^3$, the quantity required.

The value of the above method, in saving labour, will be more fully seen when the root has more than two terms, or, if numerical, more than two figures.

$$
\begin{array}{lll}
\text{Ex.} & & 8x^3 + 12x^2y + 6xy^2 + y^2 \; (2x + y \\
 & & 8x^3 \\
6x + y & 12x^2 & \quad 12x^2y + 6xy^2 + y^3 \\
 & \quad + 6xy + y^3 & \\
 & 12x^2 + 6xy + y^2 & \quad 12x^2y + 6xy^2 + y^3 \\
\end{array}
$$

Ex. **27.**

Find the cube roots

1. Of $x^3+6x^2y+12xy^2+8y^3$.
2. Of $a^3-9a^2+27a-27$.
3. Of $x^3+12x^2+48x+64$.
4. Of $8a^3-36a^2b+54ab^2-27b^3$.
5. Of $a^3+24a^2b+192ab^2+512b^3$.
6. Of $8x^3-84x^2y+294xy^2-343y^3$.
7. Of $m^3-12m^2nx+48mn^2x^2-64n^3x^3$.
8. Of $a^3x^3-15a^2bx^3+75ab^2x^3-125b^3x^3$.

---

54. If the root consist of more than two terms, as $a+b+c$, we may (just as in the case of the square root) first find $a+b$ as above, and put this $=a'$: then, at this point of the operation, we shall have subtracted first a^3 and then $3a^2b+3ab^2+b^3$, that is, altogether $(a+b)^3$ or a'^3; and therefore, since the whole quantity $(a+b+c)^3 = (a'+c)^3 = a'^3+3a'^2c+3a'c^2+c^3$, the remainder will be no other than $3a'^2c+3a'c^2+c^3$. [In fact, as was done in the case of the square root, it may be easily shewn to be identical with this.] If, therefore, we take now as trial-divisor $3a'^2$, just as we before took $3a^2$, we shall get c the third term in the root, and subtracting the quantity $3a'^2c+3a'c^2+c^3$, we shall have no more remainder.

Now the process of finding $3a'^2$ is much simplified by observing that it $=3\,(a+b)^2 = 3a^2+6ab+3b^2$; but, if we add b^2, *the square of the last term in the root*, to the two lines $\begin{matrix}(3a+b)\,b \\ 3a^2+3ab+b^2\end{matrix}$, the sum will be exactly $3a^2+6ab+3b^2$, the quantity required. By this means then we get $3a'^2$, and then have only to set to the left of it $3a'$ or $3a+3b$, (which may be found by tripling the last term of the preceding divisor $3a+b$) and proceed just as we did before when we had set down $3a$ and $3a^2$—that is, first finding c, and then forming, as before, $3a'^2c+3a'c^2+c^3$, which we subtract, making, with a'^3 already subtracted, $(a'+c)^3$ or $(a+b+c)^3$ subtracted altogether. And so on, if the root were $a+b+c+d$, &c.

The student should study carefully the first of the two following Examples, as it is the type to which all others are referred.

Ex. 1

$$a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^3 + c^3 \,(a + b + c)$$

$$a^3$$

$$3a + b \qquad 3a^2 \qquad\qquad 3a^2b + 3ab^2 + b^3$$

$$+ (3a + b)\,b$$

$$3a^2 + 3ab + b^2 \qquad 3a^2b + 3ab^2 + b^3$$

$$3a' + c = 3a + 3b + c \quad 3a'^2 = 3a^2 + 6ab + 3b^2 \qquad 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$$

$$(3a' + c)\,c = + (3a + 3b + c)\,c$$

$$3a'^2 + 3a'c + c^2 = 3a^2 + 6ab + 3b^2 + 3ac + 3bc + c^2 \qquad 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$$

Ex. 2.

$$x^6 - 3x^5 - 3x^4 + 11x^3 + 6x^2 - 12x - 8 \,(x^2 - x - 2$$

$$x^6$$

$$3x^2 - x \qquad 3x^4 \qquad\qquad - 3x^5 - 3x^4 + 11x^3$$

$$- 3x^3 + x^2$$

$$3x^4 - 3x^3 + x^2 \qquad - 3x^5 + 3x^4 - x^3$$

$$3x^2 - 3x - 2 \qquad 3x^4 - 6x^3 + 3x^2 \qquad -6x^4 + 12x^3 + 6x^2 - 12x - 8$$

$$- 6x^2 + 6x + 4$$

$$3x^4 - 6x^3 - 3x^2 + 6x - 4 \qquad -6x^4 + 12x^3 + 6x^3 - 12x - 8$$

In Ex. 2 the student should be required to mention to what parts in Ex. 1 the different quantities correspond: thus, $3x^2$ to $3a$, $3x^2 - x$ to $3a + b$, $3x^4$ to $3a^2$, $-3x^3 + x^2$ to $3ab + b^2$, $3x^4 - 3x^3 + x^2$ to $3a^2 + 3ab + b^2$, $-3x^5 + 3x^4 - x^3$ to $3a^2b + 3ab^2 + b^3$; so also $3x^2 - 3x$ to $3a'$, $3x^2 - 3x - 2$ to $3a' + c$, $3x^4 - 6x^3 + 3x^2$ to $3a'^2$, &c.

Ex. 28.

Extract the cube roots.

1. Of $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$.
2. Of $x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8$.
3. Of $a^6 - 3a^5b + 6a^4b^2 - 7a^3b^3 + 6a^2b^4 - 3ab^5 + b^6$.
4. Of $x^6 - 12ax^5 + 60a^2x^4 - 160a^3x^3 + 240a^4x^2 - 192a^5x + 64a^6$.
5. Of $8x^6 + 48x^5y + 60x^4y^2 - 80x^3y^3 - 90x^2y^4 + 108xy^5 - 27y^6$.
6. Of $x^9 - 3x^8 + 6x^7 - 10x^6 + 12x^5 - 12x^4 + 10x^3 - 6x^2 + 3x - 1$.
7. Of $a^3 - b^3 + c^3 - 3(a^2b - a^2c - ab^2 - ac^2 - b^2c + bc^2)\ 6abc$.
8. Of $1-6x+21x^2-56x^3+111x^4-174x^5+219x^6-204x^7+144x^8-64x^9$.

---

55. Since $1 = 1^3$, $1000 = 10^3$, $1000000 = 100^3$, &c., it follows that the cube root of any number between 1 and 1000, that is, having *one*, *two*, or *three* figures, is a number of *one* figure; so also the cube root of any number between 1000 and 1000000, that is, having *four*, *five*, or *six* figures, is a number of *two* figures, &c. Hence, if we set a dot over every *third* figure of any given cube number, beginning with the units-figure, the number of dots will exactly indicate the number of figures in its cube root.

If the root have any number of decimal places, the cube will have *thrice* as many; and therefore the number of decimal places in every cube decimal will be necessarily a *multiple of three*, and the number of decimal places in the root will be a third of that number. Hence, if the given cube number be a decimal, and consequently have its number of decimal places a multiple of three, by setting as before the dot upon the *units-figure*, and then over every third figure on *both* sides of it, the number of dots to the left will still indicate the number of *integral* figures in the root, and the number to the right the number of *decimal* places.

If the given number be not a perfect cube, we may dot as before, (always *setting the dot first upon the units-*

figure), and annex cyphers as in the case of the square root, so as to approximate to the cube root required, to as many decimal places as we please.

It will be seen, by the following example, how the numerical process corresponds to the algebraical. The cyphers are omitted, except that in the numbers corresponding to $3a^2$, $3a'^2$, &c., it is necessary to express two at the end: thus a is really 4000, and therefore $3a^2$ is 48000000; but as in the first remainder we only need the figures of the first and second periods, corresponding to 43 in the root, we may treat the a as 40, and thus $3a^2$ will be 4800, and $3a$ will be 120, so that $3a + b$ will become 123.

Ex.		80677568161 (4321
		64
123	4800	16677
	369	
	5169	15507
1292	554700	1170568
	2584	
	557284	1114568
12961	55987200	56000161
	12961	
	56000161	56000161

Ex. 29.

Find the cube roots of

1. 9261, 12167, 15625, 32768, 103.823, 110592, 262144, 884.736.
2. 1481544, 1601.613, 1953125, 1259712, 2.803221, 7077888.
3. 12.812904, 8741816, 56.623104, 33076.161, 22425768.
4. 102503.232, 820025856, 264.609288, 1076890625, 2.116874304.
5. Extract to 4 figures the cube roots of 2.5, .2, .01, 4.

CHAPTER V.

GREATEST COMMON MEASURE: LEAST COMMON MULTIPLE.

56. When one quantity divides another without remainder, it is said to *measure* it, and is called a *measure* of it.

Thus, 3, a, b, $3a$, ab, a^2, &c. are all measures of $3a^2b$.

A *common* measure of two quantities is one which divides each of them without remainder.

Thus, a, b, $3a$, $3b$, ab, $3ab$, are all common measures of $3a^2b$ and $15abc$; and their *greatest* common measure, that is, the largest common factor they contain, is $3ab$.

57. It is commonly easy to detect *by inspection*, *i. e.* by looking at the two quantities, their largest common measure, if it is a *simple* factor, that is, if it consists of only one term; because then it will be found as a factor in every term of each of them.

Thus, $3xy$ will divide every term of $3x^3y - 6xy^3$ and also of $3xy - 9x^2y^2$; it is therefore a *common measure* of them: and since, when these are divided by $3xy$, the quotients $x^2 - 2y^2$ and $1 - 3xy$ have no common factor, $3xy$ is their *greatest* common measure (G. C. M.).

So $2a^2b$ is the greatest divisor of $6a^2b^2 - 8a^4b$, and a^2c of $2a^2c^2 - 5a^2bc$; and a^2, which is the G. C. M. of $2a^2b$ and a^2c, is plainly therefore the G. C. M. of $6a^2b^2 - 8a^4b$ and $2a^2c^2 - 5a^2bc$.

Ex. 30.

Find the G. C. M. of

1. $3x^3$ and $12x^2y$; $4a^2b^2$ and $-6ab^3$; $-12x^2y^3z^4$ and $8y^5z^3$.
2. $3ax^2 - 2a^2x$ and $a^2x^2 - 3abx$; $3a^3 + 2a^2b - 5ab^2$ and $2a^2b + 2ab^2$; $6x^3y - 12x^2y^2 + 3xy^3$ and $4ax^2 + 4axy + 4a^2x$.

---

58. In like manner we may sometimes find by inspection the G. C. M. of two quantities, when not a simple factor, if it happens to be easy to separate them into their component factors.

Ex. 1. The G. C. M. of $6a^2x^2 (a^2 - x^2)$ and $4a^3x (a + x)^2$ is $2a^2x (a + x)$.

Ex. 2. The G. C. M. of $a^2 (a^2x^2 - 3ax^3 + 2x^4)$ and $x^2 (a^4 - 4a^2x^2)$, that is, of $a^2x^2 (a^2 - 3ax + 2x^2)$ or $a^2x^2 (a - 2x) (a - x)$ and $a^2x^2 (a^2 - 4x^2)$, is $a^2x^2 (a - 2x)$.

Ex. 31.

Find by inspection the G. C. M. of

1. $4x^2 (a+x)^2$ and $10 (a^2x-x^3)^2$.
2. $x^2 (a^2-x^2)^2$ and $(a^2x+ax^2)^3$.
3. $(a^2b-ab^2)^2$ and $ab (a^2-b^2)^2$.
4. $6 (x^2-1)$ and $8 (x^2-3x+2)$.
5. $(x^2+x)^2$ and $x^3 (x^2-x-2)$.
6. $4 (x^3+a^3)$ and $6 (x^2-2ax-3a^2)$.
7. $a^3 (x^2 + 12x + 11)$ and $a^2x^2 - 11a^2x - 12a^2$.
8. $9 (a^2x^2 - 4)$ and $12 (a^2x^2 + 4ax + 4)$.

---

59. But if the greatest common measure of two quantities be a *compound* quantity, it cannot generally be thus easily found by inspection, but may always be obtained by a method we are now about to explain, the proof of which will be given hereafter.

Def. An algebraical quautity is said to be of so many *dimensions*, as is indicated by the highest index of its letter of reference.

Thus $x^2 - 7x + 10$ is of *two* dimensions, $x^3 + 1$ of *three*.

If it also involve other letters, it is said to be of so many dimensions in each of them, according to the highest indices of each.

Thus $x^4y + 3x^3y^2 + x^2y^3$ is of *four* dimensions in x, and *three* in y.

If the dimensions of each term are the *same*, the quantity is said to be *homogeneous*, and of so many dimensions as is indicated by the *sum* of the indices in each term.

Thus the last quantity is *homogeneous*, and of *five* dimensions.

The word *dimensions* has been adopted from the language of Geometry;—such quantities as a, b, &c. being compared to *lines* (which have only *one* dimension, viz. length), and called *linear* quantities; such quantities as a^2, ab, &c. to *areas* (which have two dimensions, length and breadth); and such as a^3, a^2b, abc, &c. to *solids* (which have *three* dimensions, length, breadth, and thickness: beyond this we have no corresponding quantities in Geometry; but the term dimensions, having been once employed in Algebra, has been retained in all other cases.

60. Let there be given then two algebraical quantities, of which it is required to find the G. C. M. Arrange them according to powers of some common letter, and divide the one of higher dimensions by the other; or if the highest index happen to be the *same* in each, take either of them for dividend. Take now, as in Arithmetic, the remainder after this division for divisor, and the preceding divisor for dividend, and so on until there is no remainder: then the last divisor will be the G. C. M. of the two given quantities.

Ex. Find the G. C. M. of $x^2 - 7x + 10$ and $4x^3 - 25x^2 + 20x + 25$.

$$
\begin{array}{l}
x^2 - 7x + 10)\ 4x^3 - 25x^2 + 20x + 25\ (4x + 3 \\
\qquad\qquad\qquad\ \underline{4x^3 - 28x^2 + 40x} \\
\qquad\qquad\qquad\qquad\quad 3x^2 - 20x + 25 \\
\qquad\qquad\qquad\qquad\quad \underline{3x^2 - 21x + 30} \\
\qquad\qquad\qquad\qquad\qquad\quad x - 5)\ x^2 - 7x + 10\ (x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^2 - 5x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ - 2x + 10 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{- 2x + 10}
\end{array}
$$

Ans. $x - 5$.

We may as well observe, that the expression *Greatest* C. M., which has been adopted from Arithmetic, must be understood in Algebra as applying not to the *numerical* magnitude, positive or negative, of the quantity, but to its *dimensions* only, on which account it is sometimes called the *Highest* C. M. Thus it would be quite immaterial whether, in the above example, we consider the G. C. M. to be $x - 5$ or $5 - x$: and either of these, in fact, might be made *numerically* greater than the other, by giving different values to x.

Ex. 32.

Find the G. C. M.

1. Of $3x^2 + x - 2$ and $3x^2 + 4x - 4$.
2. Of $6x^2 + 7x - 3$ and $12x^2 + 16x - 3$.
3. Of $9x^2 - 25$ and $9x^2 + 3x - 20$.
4. Of $8x^2 + 14x - 15$ and $8x^3 + 30x^2 + 13x - 30$.
5. Of $4x^2+3x-10$ and $4x^3+7x^2-3x-15$.
6. Of $2x^4 + x^3 - 20x^2 - 7x + 24$ and $2x^4 + 3x^3 - 13x^2 - 7x + 15$.

---

61. If the given quantities have *both* or *either* of them, in any case, *simple* factors, as in (57), these must be struck out, and the Rule applied to the resulting quantities. Then the G. C. M. of these, being found as above, will be the same as that of the given ones; except it should happen that we have to strike factors out of *both* of them, and that *these factors themselves* have a common factor. In this case the G. C. M. found, as above, of the resulting quantities, must be multiplied by this common factor, in order to produce that of the given ones.

So also, whenever we convert a remainder, according to the Rule, into a divisor, we may strike out of it any simple factor it may contain. Here, however, there is no restriction, as in the former case; because no part of such a simple factor *can* be common also to the new dividend, which, being the same as the former divisor, will be already clear of simple factors. It is only with the *first* pair, or *given quantities*, that we shall have to attend to this.

And if, moreover, the first term of any such remainder is negative, we may, for the sake of neatness, before taking it as a new divisor, change the signs of all its terms, which is equivalent to dividing it by -1. This can only affect the *signs* of the G. C. M.

Ex. Find the G. C. M. of

$$2x^5 - 8x^4 + 12x^3 - 8x^2 + 2x \text{ and } 3x^5 - 6x^3 + 3x.$$

Here, striking out of the first the factor $2x$ (which is common to all its terms) and of the second the factor $3x$, we reduce the quantities to $x^4 - 4x^3 + 6x^2 - 4x + 1$ and $x^4 - 2x^2 + 1$; but as $2x$ and $3x$ have themselves a common factor, x, it is plain that the original quantities have a common factor, x, which these latter quantities have not; hence the G. C. M. of these, when found, must be multiplied by x to produce that of the given quantities.

$$\begin{array}{rl} x^4-2x^2+1) & x^4-4x^3+6x^2-4x+1 \ (1 \\ & \underline{x^4-2x^2+1} \\ -4x \,| & \underline{-4x^3+8x^2-4x} \\ & \quad x^2-2x+1 \end{array}$$

$$\begin{array}{rl} x^2-2x+1) & x^4-2x^2+1 \ (x^2+2x+1 \\ & \underline{x^4-2x^3+x^3} \\ & \quad 2x^3-3x^2+1 \\ & \quad \underline{2x^3-4x^2+2x} \\ & \qquad x^2-2x+1 \\ & \qquad \underline{x^2-2x+1} \end{array}$$

In this Example, the first remainder is reduced by dividing it by $-4x$; and, the G. C. M. of these two quantities being x^2-2x+1, that of the two given quantities will be $x(x^2-2x+1)$ or x^3-2x^2+x.

Ex. 33.

Find the G. C. M.

1. Of $a^3 + x^3$ and $a^2 + 2ax + x^2$. 2. Of $x^2 + x - 2$ and $x^2 - 3x + 2$.
3. Of $2x^3 + 6x^2 + 6x + 2$ and $6x^3 + 6x^3 - 6x - 6$.
4. Of $2y^3 - 10y^2 + 12y$ and $3y^4 - 15y^3 + 24y^2 - 24$.
5. Of $x^3 - 6ax^2 + 12a^2x - 8a^3$ and $x^4 - 4a^2x^2$.
6. Of $2x^3 + 10x^2 + 14x + 6$ and $x^3 + x^2 + 7x + 39$.
7. Of $3x^3 + 3x^2 - 15x + 9$ and $3x^4 + 3x^3 - 21x^2 - 9x$.
8. Of $x^3 + x^2y + xy^2 + y^3$ and $x^4 + x^3y + xy^3 - y^4$.
9. Of $2a^4 + a^3b - 4a^2b^2 - 3ab^3$ and $4a^4 + a^3b - 2a^2b^2 + ab^3$.
10. Of $3a^6+15a^5b-3a^3b^2-15a^2b^3$ and $10a^5-30a^4b-10a^2b^2+30ab^3$.
11. Of $x^4 - 2x^3y + 2xy^3 - y^4$ and $x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4$.
12. Of $x^4 + 6x^3 + 11x^2 + 4x - 4$ and $x^4 + 2x^3 - 5x^2 - 12x - 4$.

---

62. If now, having first attended to the directions of (61), we find, at any step of our process, that the first term of the dividend is not *exactly* divisible by the first of the divisor, then, in order to avoid fractions in the quotient, we may multiply the whole dividend by

such a simple factor, as will make its first term so divisible.

Ex. Find the G. C. M. of $6x^2y+4xy^2-2y^3$ and $8x^3+4x^2y-4xy^2$.

Stripping them of their simple factors, $2y$ and $4x$, (and noting that these contain the common factor, 2), we have $3x^2+2xy-y^2$ and $2x^2+xy-y^2$, and proceed with these quantities as follows:

$$\begin{array}{r|l}
 & 3x^2 + 2xy - y^2 \\
 & 2 \\
2x^2+xy-y^2) & 6x^2 + 4xy - 2y^2\ (3 \\
 & 6x^2 + 3xy - 3y^2 \\
 & y\,|\,xy + y^2 \\
x + y) & 2x^2 + xy - y^2\ (2x - y \\
 & 2x^2 + 2xy \\
 & -xy - y^2 \\
 & -xy - y^2
\end{array}$$

The G. C. M. then will be $2(x+y)$, it being plain that the G. C. M. of $2(3x^2+2xy-y^2)$ and $2x^2+xy-y^2$ will be the same as that of $3x^2+2xy-y^2$, and $2x^2+xy-y^2$, because the 2 introduced into the first is no factor of the second quantity.

Ex. 34.

Find the G. C. M.

1. Of $6x^2+13x+6$ and $8x^2+6x-9$.
2. Of $15x^2-x-6$ and $9x^2-3x-2$.
3. Of $6x^2-x-2$ and $21x^3-26x^2+8x$.
4. Of $6x^3-6x^2+2x-2$ and $12x^2-15x+3$.
5. Of $3x^3-22x-15$ and $5x^4+x^3-54x^2+18x$.
6. Of $3x^3-3x^2y+xy^2-y^3$ and $4x^3-x^2y-3xy^2$.
7. Of x^3-8x+3 and x^6+3x^5+x+3.
8. Of $5x^3+2x^2-15x-6$ and $-7x^3+4x^2+21x-12$.
9. Of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$.
10. Of $6x^4-x^3y-3x^2y^2+3xy^3-y^4$ and $9x^4-3x^3y-2x^2y^2+3xy^3-y^4$.
11. Of $12x^5-12x^3y^2+12x^2y^3-3xy^4$ & $12x^5+8x^4y-18x^3y^2-6x^2y^3+4xy^4$.
12. Of $x^4-2x^3+x^2-8x+8$ and $4x^3-12x^2+9x-1$.

---

63. In order to prove the rule above given, it will be necessary to show first the truth of the following statement.

If a quautity c *be a common measure of* a *and* b, *it will also measure the sum or difference of any multiples of* a *and* b, *as* ma ± nb.

For let c be contained p times in a, and q times in b; then $a = pc$, $b = qc$, and $ma \pm nb = mpc \pm nqc = (mp \pm nq)\,c$; hence c is contained $mp \pm nq$ times in $ma \pm nb$, and therefore c measures $ma \pm nb$.

Thus, since 6 will divide 12 and 18 without remainder, it will also divide any number such as $7 \times 12 + 5 \times 18$, $11 \times 12 - 3 \times 18$, 12 (or 1×12) $+ 7 \times 18$, $5 \times 12 - 18$, &c., *i.e.* any number found by adding or subtracting any multiples of 12 and 18.

64. *To prove the Rule for finding the greatest Common Measure of two quantities.*

First, let the two given quantities, denoted by a and b, have neither of them any simple factor.

Let a be that which is not of lower dimensions than the other; and suppose a divided by b, with quotient p and remainder c, b by c, with quotient q and remainder d, &c.

$b)\ a\ (p$	546) 672 (1
$\quad pb$	546
$\quad c)\ b\ (q$	126) 546 (4
$\qquad qc$	504
$\qquad d)\ c\ (r$	42) 126 (3
$\qquad\quad rd$	126

Then, by (63), all the common measures of a and b are also measures of $a - pb$ or c, and are therefore common measures of b and c; and, conversely, all the common measures of b and c are also measures of $pb + c$ or a, and are therefore common measures of a and b: hence it is plain that b and c have precisely the *same* common measures as a and b.

In like manner, it may be shewn that c and d have the same common measures as b and c, and therefore the same as a and b.

And so we might proceed if there were more remainders, the quantities a, b, c, d, &c. getting lower and lower, yet still being such that a and b, b and c, c and d, &c. have the same common measures.

But, if d divides c without remainder, then d is itself the greatest quantity that divides both c and d, that is, d is the *greatest* of the common measures of c and d, and therefore is the *Greatest Common Measure* of a and b.

Thus, in the numerical example, the common divisors of 546 and 672 are precisely the same as those of 126 and 546, and these again are the same as those of 42 and 126: but 42 is the G. C. M. of 42 and 126, and is therefore the G. C. M. of 126 and 546, and also of 546 and 672.

65. Next, let a and b have simple factors, and let $a = \alpha a'$, $b = \beta b'$, where α denotes the product of *all* the simple factors in a, and β of those in b, and a', b' are the resulting quantities, when these simple factors are struck out; then a' b', having neither of them any simple factor, will have no factor in common with α or β. Now a or $\alpha a'$ is made up only of the factors in α and a', and b or $\beta b'$ only of those in β and b'. Hence, if α be *prime* to β, (that is, *if α have no factor in common with β*), the only factors which a can have in common with b must be those which a' may have in common with b', that is, the G. C. M. of a and b will be the same as that of a' and b'. But, if α and β have any common factor, then this will also be common to a and b, besides what may be common to a' and b', that is, the G. C. M. of a and b will be obtained by multiplying the G. C. M. of a' and b' by the common factor of α and β.

Hence this case also is reduced to finding the G. C. M. of two quantities a' and b', *which have no simple factors.* And, of course, the above reasoning holds if either α or β be unity, that is, if *one* only of the given quantities have a simple factor to be struck out.

66. Having shewn that we may strike any simple factors out of the original quantities, we shall now shew that we may strike them also out of any of the remainders.

Let then a', b', represent quantities having no simple factors, (either the original quantities, a, b, if they have no simple factor, or else a, b, reduced, as above); and let us apply the Rule to a', b', dividing a' by b', and obtain the first remainder c: then we know that the G. C. M. of a' and b' is the same as that of b' and c. Suppose now that $c = \gamma c'$, where γ is a simple factor, and c' a compound quantity, having no simple factor. Then c is made up of the factors in γ and c'; and b' (having no *simple* factor) can have no factor in common with γ, and therefore can have none in common with c but such as it may have in common with c'; that is, the G. C. M. of b' and c is the same as that of b' and c'. And, of course, the same reasoning holds with the other remainders.

67. Lastly, if, at any step (supposing simple factors struck out), the first term of the dividend should not be exactly divisible by the first of the divisor, as, for instance, in the case of a' and b', we may multiply the dividend a' by any simple factor α', which will make it so divisible: for, since the divisor b' has no *simple* factor, it can have no factor in common with α', nor therefore any in common with the dividend $\alpha' a'$, but what it may have in common with a', that is, the G. C. M. of a' and b' will be the same as that of $\alpha' a'$ and b'.

68. When one quantity *contains* another, as a divisor without remainder, it is said to be a *multiple* of it; and a *common* multiple of two or more quantities is one that contains each of them without remainder.

Thus, $6x^3y$ is a common multiple of $2x^2$, $3xy$, $6x^3$, &c., and any quantity is a multiple of any of its measures.

Of course, the *least* common multiple (L. C. M.) of two or more quantities is the least quantity that can be formed, so as thus to contain each of them.

69. *To find the Least Common Multiple of two quantities.*

Let a and b represent the two quantities, d their G. C. M.; and let $a=pd$, $b=qd$, so that p and q will have no common factor. Then the least quantity which contains p and q will be pq, and therefore the least quantity which contains pd and qd will be pqd, which is consequently the L. C. M. required of a and b.

Since $pqd=\frac{pd \times qd}{d}=\frac{a \times b}{d}$, it appears that the L. C. M. of a and b may be found by dividing their product by their G. C. M.; or, which is more simple in practice, by dividing *either* of them by their G. C. M., and multiplying the quotient by the other.

The L. C. M., however, of two or more quantities is generally formed by inspection, and, with a little practice, there is no difficulty in this, as we have only to set down the factors which compose them, omitting any that may occur more than once, and the product of these will be the L. C. M. required.

Ex. 1. Find the L. C. M. of $2bx$, $6abxy$, $3acx$.
Here the factors are $2bx$, $3ay$, c; and the L. C. M. is $6abcxy$.

Ex. 2. Find the L. C. M. of $2a^2(a+x)$, $4ax(a-x)$, $6x^2(a+x)$.
Here the L. C. M. of the simple factors is $6a^2x^2$, that of the compound factors is a^2-x^2; therefore the L. C. M. required is $12a^2x^2(a^2-x^2)$.

Ex. 35.

Find the L. C. M.

1. Of $4a^2bc$ and $6ab^2c$; of $9x^3y$ and $12xy^3$; of axy and $a(xy-y^2)$; of $ab + ad$ and $ab - ad$.
2. Of $8a^4$, $10a^3b$, and $12a^2b^2$; of a^5, $5a^4b$, $10a^3b^2$, $10a^2b^3$, $5ab^4$, and b^5; of $9x^2$, $6ax$, $8a^2$, $36x^3$, $3ax^2$, $50a^2x$, and $24a^3$.
3. Of $2(a+b)$ and $3(a^2-b^2)$; of $4(a^2-a)$ and $6(a^2+a)$; of $6(x^2+xy)$, $8(xy-y^2)$, and $10(x^2-y^2)$.
4. Of $4(a^3-ab^2)$, $12(ab^2+b^3)$, $8(a^3-a^2b)$; and of $6(x^2y+xy^2)$, $9(x^3-xy^2)$, $4(y^3+xy^2)$.

---

70. *Every common multiple of* a *and* b *is a multiple of their* L. C. M.

For let M be any common multiple of a and b, and m their L. C. M.; and let M contain m (if possible) r times with remainder s, which will of course be less than the divisor m; hence we should have

$$M = rm + s, \text{ and, therefore, } s = M - rm:$$

but since a and b measure both M and m, they would also (63) measure $M - rm$, or s; i. e. s, which is less than m, would be a common multiple of a and b, contrary to our supposition that m was their *least* common multiple. Hence M will contain m with *no* remainder, and will therefore be a *multiple* of m.

CHAPTER VI.

FRACTIONS.

ALGEBRAICAL Fractions are for the most part precisely similar both in their nature and treatment to common Arithmetical Fractions. We shall have, therefore, to repeat much of what has been said in Arithmetic; but the Rules which were there *shewn* to be true only in the particular examples given, will here, by the use of letters, which stand for *any* quantities, be proved to be true in *all* cases.

71. A Fraction is a quantity which represents a part or parts of an unit or whole.

It consists of two members, the *numerator* and *denominator*, the former placed over the latter with a line between them. Now we have already agreed (8) that such an expression shall denote that the upper quantity is divided by the lower; and, in accordance with this, it will be seen presently that a fraction does also express the quotient of the num^r divided by the den^r.

The den^r shews into how many equal parts the unit is divided, and the num^r the number taken of such parts.

Thus $\frac{a}{b}$ means that the unit is divided into b equal parts, a of which are taken.

Every integral quantity may be considered as a fraction whose den^r is 1; thus a is $\frac{a}{1}$.

72. To *multiply* a fraction by an integer, we may either multiply the num^r or divide the den^r by it; and, conversely, to *divide* a fraction by any integer, we may either divide the num^r or multiply the den^r by it.

Thus $\frac{a}{b} \times x = \frac{ax}{b}$; for in each of the fractions $\frac{a}{b}, \frac{ax}{b}$, the unit is divided into b equal parts, and x times as many of them are taken in the latter as in the former; hence the latter fraction is x times the former, that is $\frac{ax}{b} = \frac{a}{b} \times x$: and, by similar reasoning, $\frac{ax}{b} \div x = \frac{a}{b}$.

Again $\frac{a}{b} \div x = \frac{a}{bx}$; for in each of the fractions $\frac{a}{b}, \frac{a}{bx}$, the same number of parts is taken, but each of the parts in the latter is $\frac{1}{x}$th of each in the former, since the unit in the latter case is divided into x times as many parts as in the former; hence the latter fraction is $\frac{1}{x}$th of the former, that is, $\frac{a}{bx} = \frac{a}{b} \div x$: and, similarly, $\frac{a}{bx} \times x = \frac{a}{b}$.

73. If any quantity be *both* multiplied and divided by the same quantity, its value will, of course, remain unaltered. Hence if the num^r and den^r of a fraction be *both* multiplied or divided by the same quantity, its value will remain unaltered.

$$\text{Thus } \frac{a}{b} = \frac{ax}{bx} = \frac{a^2}{ab} = \&c., \text{ and } \frac{a^3b}{a^2bc} = \frac{a}{c} = \frac{ac}{c^2} = \&c.$$

74. Since $a = \frac{a}{1}$ (71), and, therefore a divided by b $= \frac{a}{1} \div b = \frac{a}{b}$ (72), it follows, as stated in (71), that a fraction represents the quotient of the num^r by the den^r. In fact, we may get $\frac{1}{b}$th of a units, (or $a \div b$,) by taking $\frac{1}{b}$th part of *each* of the a units, and this is the same as a such parts of one unit, which (71) is expressed by $\frac{a}{b}$.

Hence it is that, in Arithmetic, $\frac{1}{4}$ of £3 is the same as $\frac{3}{4}$ of £1, &c.

75. To reduce an integer to a fraction with a given denominator, multiply it by the given denominator, and the product will be the numerator of the required fraction.

Thus a, expressed as a fraction with den^r x, is $\frac{ax}{x}$; or, with den^r $b-c$, is $\frac{ab-ac}{b-c}$.

The truth of this is evident from (73).

76. The signs of all the terms in both the num^r and den^r of a fraction may be changed without altering its value: thus $\frac{x^2-2ax-a^2}{3ax-x^2}$ is identical with $\frac{a^2+2ax-x^2}{x^2-3ax}$.

This follows also from (73), as the process is equivalent to that of multiplying both num^r and den^r by -1.

77. To reduce a fraction to its lowest terms, divide the numerator and denominator by their G. C. M.

Ex. 1. $\frac{a^2x^2y^2}{a^2xy+axy^2}=\frac{a^2x^2y^2}{axy(a+y)}=\frac{axy}{a+y}$.

Ex. 2. $\frac{a^3+x^3}{a^2-x^2}=\frac{(a+x)(a^2-ax+x^2)}{(a+x)(a-x)}=\frac{a^2-ax+x^2}{a-x}$.

Ex. 3. $\frac{x^2+4x+3}{x^2+5x+6}=\frac{(x+3)(x+1)}{(x+3)(x+2)}=\frac{x+1}{x+2}$.

Ex. 4. $\frac{x^3+x^2+3x-5}{x^2-4x+3}=\frac{(x-1)(x^2+2x+5)}{(x-1)(x-3)}=\frac{x^2+2x+5}{x-3}$.

Of course, the student should consider for a moment whether he cannot obtain the G. C. M. as in (58) by mere inspection.

Ex. 36.

Reduce to their lowest terms

1. $\frac{axy+xy^2}{axy}$, $\frac{cx+x^2}{a^2c+a^2x}$, $\frac{11m^2+22mx}{33(m^2-4x^2)}$, $\frac{14x^2-7xy}{10ax-5ay}$, $\frac{5a^3b-15a^2b^2}{20ab^3+10a^2b^2}$.

2. $\frac{6x^3-18xy^2}{6x^2y-12xy^2}$, $\frac{4m^2n^2}{2m^2n+2mn^2}$, $\frac{3a^2b^2c^2}{a^2bc+ab^2c+abc^2}$, $\frac{9x^2y^3-15xy^4}{12x^2y^2-21xy^3}$.

3. $\frac{abc+9bc-5c^2}{2abdf+18bdf-10cdf}$, $\frac{ac+by+ay+bc}{af+2bx+2ax+bf}$, $\frac{acx^2+(ad-bc)x-bd}{a^2x^2-b^2}$.

4. $\dfrac{x^2-1}{ax+x}$, $\dfrac{x^4-a^4}{x^5-a^2x^3}$, $\dfrac{a^6-b^6}{a^4-b^4}$, $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$, $\dfrac{a^2-ab+ax-bx}{a^2+ab+ax+bx}$.

5. $\dfrac{x^2-4x+3}{x^2-2x-3}$. 6. $\dfrac{x^2+2x-3}{x^2+5x+6}$. 7. $\dfrac{a^2-ab-2b^2}{a^2-3ab+2b^2}$.

8. $\dfrac{6a^2-13ax+6x^2}{10a^2-9ax-9x^2}$. 9. $\dfrac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$. 10. $\dfrac{x^2+x-12}{x^3-5x^2+7x-3}$.

11. $\dfrac{7x^2-23xy+6y^2}{5x^3-18x^2y+11xy^2-6y^3}$. 12. $\dfrac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$.

13. $\dfrac{x^3+3x^2-4}{x^3-1}$. 14. $\dfrac{x^3-3x+2}{x^3+4x^2-5}$.

15. $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$. 16. $\dfrac{3a^2x^4-2ax^2-1}{4a^3x^6-2a^2x^4-3ax^2+1}$.

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78. If the num^r be of lower dimensions than the den^r, the fraction may be considered in the light of a *proper* fraction in Arithmetic; if greater, in that of an *improper* fraction, which may be reduced to a *mixed fraction*, by dividing the num^r by the den^r, as far as the division is possible, and annexing to the quotient the remainder and divisor in the form of a fraction.

Conversely, a *mixed* fraction may be reduced to an *improper* fraction, by a process similar to that employed in Arithmetic.

Ex. 1. $\dfrac{3x^2+2x+1}{x+4} = 3x-10+\dfrac{41}{x+4}$.

Ex. 2. $x^2+x+1+\dfrac{2}{x-1} = \dfrac{x^3+1}{x-1}$.

Ex. 37.

1. Reduce to mixed fractions

$\dfrac{3x^2+6x+5}{x+4}$, $\dfrac{a^2-ax+x^2}{a+x}$, $\dfrac{2x^2+5}{x-3}$, $\dfrac{10a^2-17ax+10x^2}{5a-x}$, $\dfrac{16(3x^2+1)}{4x-1}$.

2. Reduce to improper fractions

$x^2-3x-\dfrac{3x(3-x)}{x-2}$, $a^2-2ax+4x^2-\dfrac{6x^3}{a+2x}$, $x-a+y+\dfrac{a^2-ay+y^2}{x+a}$.

Shew that

3. $1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a + b + c)(a + b - c)}{2ab}$, and

$1 - \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a - b + c)(b - a + c)}{2ab}$.

4. $a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2 = \frac{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}{4b^2}$.

---

79. To reduce fractions to a common den^r, multiply the num^r of each fraction by all the den^{rs} except its own, for the new num^r corresponding to that fraction, and all the den^{rs} together for the common den^r.

The truth of this rule is evident; since, the numerator and denominator of each fraction being *both* multiplied by the same quantities, viz. the denominators of the other fractions, its value will not be altered, though all the fractions will now appear with the same denominator.

Ex. Reduce $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}$, to a common denominator.

For the num^{rs} $a \times c \times d = acd$
$b \times b \times d = b^2d$
$c \times b \times c = bc^2$

For the den^r $b \times c \times d = bcd$;

and the required fractions are $\frac{acd}{bcd}, \frac{b^2d}{bcd}, \frac{bc^2}{bcd}$.

80. If, however, the original den^{rs} of the fractions have, any of them, common factors, this process will not give them with their *least* common den^r, which, as in Arithmetic, will be found by forming the L. C. M. of the given den^{rs}: and the num^r corresponding to any one of the given fractions will be obtained, by multiplying its numerator by that factor, which is obtained by dividing the L. C. M. by its denominator.

Ex. Reduce $\frac{a}{2bc}, \frac{c}{6abxy}, \frac{b}{3acx}$ to a common denominator.

Here the L. C. M. of the denominators being $6abcxy$, the fractions required are $\frac{3a^2cy}{6abcxy}, \frac{c^2}{6abcxy}, \frac{2b^2y}{6abcxy}$.

Ex. 38.

Reduce to common denominators,

1. $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$; $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$; $\frac{2x^2y}{3a^3}, \frac{3x^3}{4a^2b}, \frac{4y^3}{5ab^2}, \frac{5xy^2}{6b^3}$.

2. $\frac{x^2}{a^2+b^2}, \frac{y^2}{a^2-b^2}$; $\frac{a+x}{a-x}, \frac{a-x}{a+x}$; $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}$.

3. $\frac{1}{4a^3(a+x)}, \frac{1}{4a^3(a-x)}, \frac{1}{2a^2(a^2-x^2)}$.

---

81. To add or subtract fractions, reduce them to common den^{rs}, and add or subtract the num^{rs} for a new num^{r}, retaining a common den^{r}.

Ex. 1. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{bcx+acy+abz}{abc}$.

Ex. 2. Add $\frac{1+x}{1+x+x^2}+\frac{1-x}{1-x+x^2}$.

Ans. $\frac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)}=\frac{2}{1+x^2+x^4}$.

Ex. 3. From $\frac{1+x}{1+x+x^2}$ take $\frac{1-x}{1-x+x^2}$.

Ans. $\frac{(1+x)(1-x+x^2)-(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)}=\frac{2x^3}{1+x^2+x^4}$.

Ex. 4. Find the value of $2+\frac{a^2+b^2}{a^2-b^2}-\frac{a-b}{a+b}$.

Ans. $\frac{2(a^2-b^2)+(a^2+b^2)-(a-b)(a-b)}{a^2-b^2}=\frac{2a^2+2ab-2b^2}{a^2-b^2}$.

Ex. 39.

Find the value of

1. $\frac{a}{2b}-\frac{(a-b)}{2(a+b)}, \frac{a}{2b}+\frac{(a+b)}{3(a-b)}, \frac{3a-4b}{2}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}$.

2. $\frac{a^2}{a-b}-a, \frac{a}{a+b}+\frac{b}{a-b}, \frac{a}{a-b}-\frac{b}{a+b}, \frac{a-b}{a+b}+\frac{ab}{a^2-b^2}$.

3. $\frac{a}{c}-\frac{(ad-bc)x}{c(c+dx)}, \frac{a^2+b^2}{a^2-b^2}\pm\frac{a-b}{a+b}, \frac{2x^2-2xy+y^2}{x^2-xy}-\frac{x}{x-y}$.

4. $\frac{1}{2(a-x)}+\frac{1}{2(a+x)}+\frac{a}{a^2+x^2}, \frac{a-b}{ab}+\frac{c-a}{ac}+\frac{b-c}{bc}$.

5. $\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2}$.

6. $\frac{1}{2a+b} + \frac{1}{2a-b} - \frac{3a}{4a^2-b^2}$.

7. $\frac{a}{b} - \frac{(a^2-b^2)x}{b^2} + \frac{a(a^2-b^2)x^2}{b^2(b+ax)}$

8. $\frac{1}{x^2} - \frac{1}{(x^2+1)^2} + \frac{x-1}{x^2+1}$.

9. $\frac{x^2+y^2}{x^2-y^2} - \frac{y}{x-y} + \frac{x}{x+y}$.

10. $1 - \frac{x-y}{x+y} + \frac{y^2}{x^2-y^2} + \frac{2xy}{x^2+y^2}$.

11. $\frac{x}{a^2} + \frac{a-x}{a(a+x)}$.

12. $2 - \frac{x^2-y^2}{x^2+y^2} + \frac{x^2+y^2}{x^2-y^2}$.

13. $\frac{x}{a^2} - \frac{a+x}{a(a-x)}$.

14. $\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^2y-x^3}{x^2y-y^3}$.

15. $\frac{x}{1-x} - \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3}$.

16. $\frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}$.

---

82. To multiply one fraction by another, multiply the numerators together for a new numerator, and the denominators for a new denominator.

Suppose that we have to multiply $\frac{a}{b}$ by $\frac{c}{d}$:

let $\frac{a}{b} = x$, $\frac{c}{d} = y$; $\therefore a = bx$, $c = dy$, and $ac = bdxy$;

hence, (dividing each of these equals by bd), $\frac{ac}{bd} = xy$;

but $xy = \frac{a}{b} \times \frac{c}{d}$, and $\frac{ac}{bd} = \frac{a \times c}{b \times d} = \frac{\text{product of num}^{\text{rs}}}{\text{product of den}^{\text{rs}}}$

whence the truth of the rule is manifest.

Similarly we may proceed for any number of fractions.

Ex. $\frac{a+b}{c+d} \times \frac{a-d}{c-d} \times \frac{3}{2} = \frac{3(a+b)(a-b)}{2(c+d)(c-d)} = \frac{3(a^2-b^2)}{2(c^2-d^2)}$.

83. To divide one fraction by another, invert the divisor and proceed as in Multiplication.

Suppose that we have to divide $\frac{a}{b}$ by $\frac{c}{d}$:

let $\frac{a}{b} = x$, $\frac{c}{d} = y$; $\therefore a = bx$, $c = dy$;

hence, $ad = bdx$, $bc = bdy$, and $\frac{ad}{bc} = \frac{bdx}{bdy} = \frac{x}{y}$;

$$\text{but } \frac{x}{y} = x \div y = \frac{a}{b} \div \frac{c}{d}, \text{ and } \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c},$$

whence the truth of the rule is manifest.

Ex. $\frac{2a+3b}{c+d} \div \frac{c-d}{2a-3b} = \frac{(2a+3b)(2a-3b)}{(c+d)(c-d)} = \frac{4a^2-9b^2}{c^2-d^2}.$

In mult^n and div^n of fractions, it is always advisable, before multiplying out the factors of the new num^r and den^r, to see if some of them do not exist in *both* the num^r and den^r, in which case they may be struck out, and the result will be more simple.

Ex. 1. $\frac{a}{bx} \times \frac{cx}{d} = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$

Ex. 2. $\frac{5ax}{3cy} \times \frac{xy+y^2}{x^2-xy} = \frac{5a(x+y)}{3c(x-y)} = \frac{5ax+5ay}{3c(x-y)}.$

Ex. 3. $\frac{4ax}{3by} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2-ax} = \frac{4x(a+x)}{3y(c-x)} = \frac{4ax+4x^2}{3y(c-x)}.$

Ex. 4. $\frac{x^2+xy}{x-y} \div \frac{x^4-y^4}{(x-y)^2} = \frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{x^4-y^4} = \frac{x}{x^2+y^2}.$

The student should leave the *denominators* of fractions with their factors *unmultiplied*, as in Ex. 2 and 3; unless they happen to combine very simply, as $(a+x)(a+x)^2$ into $(a+x)^3$, or $(a+x)(a-x)$ into a^2-x^2. The convenience of this will be found in practice.

Ex. 40.

Find the value of

1. $\frac{2x}{a} \times \frac{3ab}{c} \times \frac{3ac}{2b}$, $\frac{ax}{(a-x)^2} \times \frac{a^2-x^2}{ab}$, $\frac{a}{bx} \times \left(b+\frac{bx}{a}\right) \times \left(1-\frac{a}{a+x}\right)$.

2. $\left(a-\frac{x^2}{a}\right)\left(\frac{a}{x}+\frac{x}{a}\right)$, $\frac{a^3-x^3}{a^3+x^3} \times \frac{(a+x)^2}{(a-x)^2}$, $\frac{2a(x^2-y^2)^2}{cx} \times \frac{x^3}{(x-y)(x+y)^2}$.

3. $\frac{a^2+2ab}{a^2+4b^2} \times \frac{ab-2b^2}{a^2-4b^2}$, $\frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{x^4-y^4}$, $\left(a^4-\frac{a^2}{x^2}\right) \times \frac{a^2x^2+abx^2}{ax+1} \times \frac{ax}{a^2-b^2}$.

4. $\frac{a^3+b^3}{a^2-b^2} \div \frac{a-b}{a+b}$, $\frac{x^3+y^3}{x^2-y^2} \div \frac{x^2-xy+y^2}{x-y}$, $\left(1+\frac{1}{x}\right) \div \left(x-\frac{1}{x}\right) \times \left(1-\frac{1}{x}\right)^2$.

5. $\left(1-\frac{b^4}{a^4}\right) \div \left(\frac{a}{b}+\frac{b}{a}\right)$, $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \frac{2ab-2b^2}{3} \times \frac{a^2+ab}{a-b}$.

6. $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b} \times \frac{x^5-b^2x^3}{x^3+b^3} \div \frac{x^4-2bx^3+b^2x^2}{x^2-bx+b^2}$.

84. A *complex* fraction, *i.e.* one in which the num^r, or den^r, or both, are fractions, may be simplified as follows.

Ex. 1. $$\frac{1-\frac{x}{2}}{4x}=\frac{\frac{2-x}{2}}{\frac{4x}{1}}=\frac{2-x}{2}\times\frac{1}{4x}=\frac{2-x}{8x}.$$

Hence observe that, when a complex fraction is put into the form of a $\frac{\text{fraction}}{\text{fraction}}$, the simple expression for it will be found by taking the product of the upper and lower quantities, or *extremes*, for the num^r, and that of the two middle ones, or *means*, for the den^r; and that any factor may be struck out from either of the extremes, if it be struck out also from one or other of the means.

Ex. 2. $$\frac{2x}{x-\frac{1}{3}}=\frac{\frac{2x}{1}}{\frac{3x-1}{3}}=\frac{6x}{3x-1}.$$

Ex. 3. $$\frac{5-\frac{1}{4}x}{x+1\frac{1}{3}}=\frac{\frac{20-x}{4}}{\frac{3x+4}{3}}=\frac{60-3x}{4(3x+4)}.$$

Ex. 4. $$\frac{x+2}{1-x+\frac{1}{2-x+\frac{3}{x}}}=\frac{x+2}{1-x+\frac{x}{2x-x^2+3}}=\frac{(x+2)(2x-x^2+3)}{(1-x)(2x-x^2+3)+x}=\frac{6+7x-x^3}{3-3x^2+x^3}.$$

Simplify

Ex. **41.**

1. $\frac{2-\frac{3}{2}x}{5}$, $\frac{x}{5-\frac{2}{3}x}$, $\frac{3-x}{x+2\frac{1}{2}}$, $\frac{3x+2\frac{1}{3}}{3\frac{1}{2}}$, $\frac{2\frac{1}{4}-\frac{1}{3}x}{\frac{2}{3}x-1\frac{1}{2}}$, $\frac{x-3\frac{1}{3}}{2\frac{3}{4}-\frac{1}{6}x}$.

2. $\frac{x-\frac{5}{2}(3x-2)}{3}$, $\frac{6x-\frac{3}{4}(3+5x)}{2\frac{1}{4}}$, $\frac{2\frac{1}{3}-\frac{1}{2}(x-2)}{\frac{1}{3}(x+1)-4\frac{1}{2}}$, $\frac{1\frac{4}{5}-\frac{2}{3}(x+2)}{\frac{3}{10}(x+1)}$.

3. $\dfrac{\frac{1}{1+x}}{1-\frac{1}{1+x}}$, $\dfrac{\frac{1}{1+x}+\frac{x}{1-x}}{\frac{1}{1-x}-\frac{x}{1+x}}$, $\dfrac{\frac{a^2+b^2}{2a^2}-\frac{2b^2}{a^2+b^2}}{\frac{a^2+b^2}{2b^2}-\frac{2a^2}{a^2+b^2}}$, $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}$.

4. $\dfrac{x-\frac{x-y}{1+xy}}{1+\frac{x(x-y)}{1+xy}}$, $\dfrac{1}{x-1+\frac{1}{1+\frac{x}{4-x}}}$, $\dfrac{x}{1-\frac{x}{1+x+\frac{x}{1-x+x^2}}}$.

85. The following results should be noticed.

If $\frac{a}{b}=\frac{c}{d}$, then

$1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, or $\frac{b}{a}=\frac{d}{c}$ (i), $\frac{a}{b}\times\frac{b}{c}=\frac{b}{c}\times\frac{c}{d}$, or $\frac{a}{c}=\frac{b}{d}$ (ii);

$\frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$ (iii), $\frac{a}{b}-1=\frac{c}{d}-1$, or, $\frac{a-b}{b}=\frac{c-d}{d}$ (iv).

$$\text{hence } \frac{a\pm b}{b}\times\frac{b}{a}=\frac{c\pm d}{d}\times\frac{d}{c}, \text{ or } \frac{a\pm b}{a}=\frac{c\pm d}{c} \text{ (v)},$$

$$\text{and } \frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}, \text{ or } \frac{a+b}{a-b}=\frac{c+d}{c-d} \text{ (vi)}:$$

and any of these last may be *inverted* by (i), or *alternated* by (ii); thus $\frac{a}{a\pm b}=\frac{c}{c\pm d}$, $\frac{a}{c}=\frac{a\pm b}{c\pm d}$, $\frac{a+b}{c+d}=\frac{a-b}{c-d}$, &c.

So that, *If any two fractions are equal, we may combine by Addition or Subtraction, in any way, the num^r and den^r of the one, provided that we do the same with the other.*

86. The above results may be yet further generalized.

For, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{m}{n}\times\frac{a}{b}=\frac{m}{n}\times\frac{c}{d}$, or $\frac{ma}{nb}=\frac{mc}{nd}$;

and, therefore, by what has been above shewn,

$\frac{ma\pm nb}{ma}=\frac{mc\pm nd}{nc}$, whence $\frac{ma\pm nb}{a}=\frac{mc\pm nd}{c}$, and $\frac{ma\pm nb}{pa}=\frac{mc\pm nd}{pc}$;

so also $\frac{ma\pm nb}{pb}=\frac{mc\pm nd}{pd}$, $\frac{ma\pm nc}{pa}=\frac{mb\pm nd}{pb}$, $\frac{ma\pm nc}{pc}=\frac{mb\pm nd}{pd}$.

$$\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc+nd}, \quad \frac{ma+nb}{mc+nd}=\frac{ma-nb}{mc-nd}, \text{ \&c.}$$

$$\text{Again, since } \frac{ma\pm nb}{a}=\frac{mc\pm nd}{c}, \text{ and } \frac{pa\pm qb}{a}=\frac{pc\pm qd}{c},$$

$$\therefore \frac{ma\pm nb}{pa\pm qb}=\frac{mc\pm nd}{pc\pm qd}, \text{ \&c.}$$

Hence we see that the statement of (85) is true of *any multiples whatever* of the numerators and denominators of the fractions.

87. Further, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2}{b^2}=\frac{c^2}{d^2}$, $\frac{a^3}{b^3}=\frac{c^3}{d^3}$, &c. $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

Hence the previous results hold with a^n, b^n, c^n, d^n, instead of a, b, c, d.

88. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a}{b}=\frac{a+c+e}{b+d+f}=\frac{ma+nc+pe}{mb+nd+pf}$.

For let $\frac{a}{b}=x=\frac{c}{d}=\frac{e}{f}$; then $a=bx,\ c=dx,\ e=fx$;

$\therefore a+c+e=bx+dx+fx=(b+d+f)x$; $\therefore x$ or $\frac{a}{b}=\frac{a+c+e}{b+d+f}$.

again, $ma=mbx,\ nc=ndx,\ pe=pfx$;

$\therefore ma+nc+pe=(mb+nd+pf)x$, and x or $\frac{a}{b}=\frac{ma+nc+pe}{mb+nd+pf}$.

So also $\frac{a_n}{b^n}=\frac{c^n}{d^n}=\frac{e^n}{f^n}$, $\therefore \frac{a^n}{b^n}=\frac{a^n+c^n+e^n}{b^n+d^n+f^n}=\frac{ma^n+nc^n+pe^n}{mb^n+nd^n+pf^n}$.

N. B. The above method of proof will evidently serve, whatever be the number of equal fractions.

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89. We know by Division that the fraction

$$\frac{1}{1-x}=1+x+x^2+x^3+\text{\&c.}+x^{n-1}+\frac{x_n}{1-x};$$

so that $\frac{x^n}{1-x}$ will be the difference between $\frac{1}{1-x}$ and the first n terms of the series: and this difference if x be < 1, becomes less and less by increasing n, that is, by taking more terms of the series; whereas, if x be > 1, it becomes greater and greater. Hence, when $x < 1$, the fraction $\frac{1}{1-x}$ expresses approximately, and with more and more of accuracy, according as we take more terms, the value of the series $1+x+x^2+$ &c.; whereas, when $x > 1$, it does not at all express the value of the series, unless we take account also of the *remainder* $\frac{x^n}{1-x}$.

Thus, if $x=\frac{1}{2}$, we have $\frac{1}{1-\frac{1}{2}}$ or $2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ &c., the sum of which series approaches more and more nearly to 2 as its *Limit*, without ever actually reaching it. But if $x=2$, we have $\frac{1}{1-2}$ or $-1=1+2+4+8+$ &c., the sum of which series departs more and more from -1: the error, however, will be corrected, if we introduce the remainder at any step; thus

$$1+2+4+\frac{8}{1-2}=7-8=-1.$$

In all such cases we may consider the sign = as expressing, not the actual equality of the two quantities, but merely that the fraction can be made to assume the form of the series, and therefore may be used as *an abridgment* for it.

90. If $x = 1$ in the above, then $\frac{1}{1-1} = 1 + 1 +$ &c., that is $\frac{1}{0} =$ an infinite number of units, which is, of course, an infinitely great quantity, and is denoted by ∞ (read *infinity*.)

The meaning of this result may be thus explained. If $x = 1$ *very nearly*, so that $1 - x$ is *very small*, then $\frac{1}{1-x}$ will be, of course, very great, and may be made *as great as we please* by still further diminishing $1 - x$, that is, by taking x still more nearly $= 1$. When, therefore, we write $\frac{1}{0} = \infty$, we are not to suppose the denominator actually *zero* (in which case the division by which we obtained the series would be absurd), but only a very small quantity; and by using the sign ∞, we mean that there is no Limit to the magnitude which the fraction $\frac{1}{1-x}$ may be made to attain, by sufficiently diminishing the denominator.

In the same sense, we may say that $\frac{a}{0} = \infty$, where a represents any finite quantity whatever.

CHAPTER VII.

SIMPLE EQUATIONS CONTINUED.

91. The following equations, involving algebraical fractions, may now easily be solved, by help of the preceding chapter, after the methods in Ex. **18, 19**.

Ex. 42.

1. $a - \frac{bx}{a} = \frac{ax - b^2}{c}$. 2. $\frac{a}{x} + \frac{b}{c} - \frac{d}{e} = 0$. 3. $\frac{x}{a} + \frac{x}{b} = c$.

4. $\frac{ax + b}{c} - \frac{a}{b} = \frac{cx + d}{e}$. 5. $\frac{a(d^2 + x^2)}{dx} = ac + \frac{ax}{d}$.

6. $\frac{a}{bx} - \frac{b}{ax} = a^2 - b^2$. 7. $\frac{ax}{b} + \frac{cx}{f} = gx + \frac{1}{f}(fh - cx)$.

8. $\frac{1}{3}\{4a(1 + x) - \frac{9}{4}(a - x)\} = \frac{1}{4}\{3a(1 - x) - \frac{16}{3}(a + x)\}$.

9. $\frac{2x + 3}{4} + \frac{4x}{3} = \frac{1}{x} + \frac{6x + 2}{3} - \frac{x + 1}{6}$.

10. $1 - \frac{x}{2}\left(1 - \frac{3}{4x}\right) = \frac{2}{3}\left(3 - \frac{5x}{2}\right) + 5\tfrac{13}{40}$.

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92. Complex fractions in an equation should first be reduced by (84); and if, in any case, the denominators contain both *simple* and *compound* factors, it is best to get rid of the *simple* factors first, and then of each compound factor in turn, observing to simplify as much as possible after each multiplication.

Ex. $\frac{25 - \frac{1}{3}x}{x + 1} + \frac{16x + 4\frac{1}{5}}{3x + 2} = 5 + \frac{23}{x + 1}$.

Here, first simplifying the complex fractions, we get

$$\frac{75 - x}{3(x + 1)} + \frac{80x + 21}{5(3x + 2)} = 5 + \frac{23}{x + 1}:$$

then, multiplying by 15, $\frac{375 - 5x}{x + 1} + \frac{240x + 63}{3x + 2} = 75 + \frac{345}{x + 1}$;

$\therefore$, mult. by $x + 1$, $375 - 5x + \frac{240x^2 + 303x + 63}{3x + 2} = 75x + 75 + 345$;

$\therefore$, simplifying, $\frac{240x^2 + 303x + 63}{3x + 2} = 75x + 5x + 75 + 345 - 375 = 80x + 45$;

$\therefore 240x^2 + 303x + 63 = 240x^2 + 295x + 90$, and $8x = 27$, or $x = 3\frac{3}{8}$.

Ex. 43.

1. $\dfrac{x+\frac{4}{5}}{7} - \dfrac{x}{2} = \dfrac{4}{35x} - \dfrac{5x}{14}.$

2. $\dfrac{2x}{3} - \dfrac{1-\frac{1}{2}x}{4x} = \dfrac{x-1}{2} + \dfrac{x}{6}.$

3. $\dfrac{8x+5}{14} + \dfrac{7x-3}{6x+2} = \dfrac{4x+6}{7}.$

4. $\dfrac{2(4x+3)}{x+3} + \dfrac{3}{x+1} = 8.$

5. $\dfrac{ax}{b(x+c)} + \dfrac{bx}{a(x+c)} = 1.$

6. $\dfrac{x-3}{x+2} = \dfrac{1}{2} + \dfrac{x-3}{2x-1}.$

7. $\dfrac{6x+a}{4x+b} = \dfrac{3x-b}{2x-a}.$

8. $\dfrac{x-\frac{1}{3}(x-1)}{3} + \dfrac{31}{36} = \dfrac{3-\frac{1}{4}(x-2)}{5}.$

9. $\dfrac{3-4x}{3(3-x)} + \dfrac{1}{2(1-x)} = 1\frac{1}{1}.$

10. $\dfrac{1}{ab-ax} + \dfrac{1}{bc-bx} = \dfrac{1}{ac-ax}.$

11. $\dfrac{(2x+3)x}{2x+1} + \dfrac{1}{3x} = x+1.$

12. $\dfrac{2x+a}{3(x-a)} + \dfrac{3x-a}{2(x+a)} = 2\frac{1}{6}.$

13. $\frac{1}{4}\{3x - \frac{2}{3}(1+x)\} + \dfrac{1-\frac{1}{5}x}{5\frac{1}{2}} = \dfrac{2\frac{2}{5} + \frac{1}{25}(x-1)}{2\frac{1}{5}}.$

14. $\dfrac{x}{a+x} = \dfrac{a+x}{x} - \dfrac{2a-b}{2x}$

15. $\dfrac{x+4}{3x+5} + 1\frac{1}{6} = \dfrac{3x+8}{2x+3}.$

16. $\frac{1}{25}(11x-13) + \frac{1}{7}(19x+3) - \frac{1}{4}(5x-25\frac{1}{3}) = 28\frac{1}{7} - \frac{1}{21}(17x+4).$

17. $\dfrac{10x+17}{18} - \dfrac{12x+2}{13x-16} = \dfrac{5x-4}{9}.$

18. $\dfrac{x+1\frac{1}{2}}{3} - \dfrac{10-x}{3\frac{2}{3}} = \dfrac{4-2\frac{2}{3}x}{11} - \dfrac{1}{11}.$

19. $\frac{1}{2}(x - 1\frac{25}{26}) - \frac{1}{13}(2-6x) = x - \frac{1}{39}\{5x - \frac{1}{4}(10-3x)\}.$

20. $\dfrac{6x+13}{15} - \dfrac{3x+5}{5x-25} = \dfrac{2x}{5}.$

21. $\dfrac{x-7}{x+7} = \dfrac{2x-15}{2x-6} - \dfrac{1}{2(x+7)}.$

22. $\dfrac{132x+1}{3x+1} + \dfrac{8x+5}{x-1} = 52.$

23. $\dfrac{7x+1}{x-1} = \dfrac{35}{9} \cdot \dfrac{x+4}{x+2} + 3\frac{1}{9}.$

24. $\dfrac{17}{6x+17} - \dfrac{10}{3x-10} = \dfrac{1}{1-2x}.$

25. $\dfrac{11}{12x+11} + \dfrac{5}{6x+5} = \dfrac{7}{4x+7}.$

26. $\frac{1}{2}x - \dfrac{\frac{1}{3}(2x-3) - \frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)} = \dfrac{3}{2} \cdot \dfrac{x^2 - \frac{1}{3}x + 2}{3x-2}$

27. $\frac{1}{10}(7x+6\frac{1}{2}) + \frac{1}{12}\{11x - \frac{1}{2}(x-1\frac{1}{2})\} = \frac{1}{5}(3x+1) + \frac{1}{22}\{43x - \frac{1}{2}(3-8x)\}.$

28. $\dfrac{6x-7\frac{1}{3}}{13-2x} + 2x + \dfrac{1+16x}{24} = 4\frac{5}{12} - \dfrac{12\frac{5}{8}-8x}{3}.$

29. $4x - \frac{1}{2}(x-2) - [2x - (\frac{1}{4}x - \frac{1}{18}\{16 - \frac{1}{2}(x+4)\})] = \frac{3}{2}(x+2).$

30. $\dfrac{6-5x}{15} - \dfrac{7-2x^2}{14(x-1)} = \dfrac{1+3x}{21} - \dfrac{2x-2\frac{1}{5}}{6} + \dfrac{1}{105}.$

93. The following are additional Problems in Simple Equations, presenting somewhat more of difficulty than those given under (41).

Ex. 1. A fish was caught whose tail weighed 9 lbs; his head weighed as much as his tail and half his body; and his body weighed as much as his head and tail. What did the fish weigh?

It is sometimes convenient to take x to represent, not the quantity actually demanded in the question, but some other unknown quantity on which this one depends. It is only experience, however, and practice which can suggest these cases; but this example is one of them.

Let x = weight of body;
$\therefore 9 + \frac{1}{2}x$ = weight of tail + $\frac{1}{2}$ body = weight of head;
but the body weighs as much as head and tail;
$\therefore x = (9 + \frac{1}{2}x) + 9$, whence $x = 36$, weight of body;
$\therefore 9 + \frac{1}{2}x = 27$, weight of head;
and the whole fish weighed $27 + 36 + 9 = 72$ lbs.

Ex. 2. A gamester at one sitting lost $\frac{1}{5}$ of his money, and then won 10*s*; at a second he lost $\frac{1}{3}$ of the remainder, and then won 3*s*; and now he has 3 guineas left. How much money had he at first?

Let x = number of shillings he had at first;
having lost $\frac{1}{5}$ of it, he had $\frac{4}{5}$ of it, or $\frac{4}{5}x$ remaining;
he then won 10*s*, and had, therefore, $\frac{4}{5}x + 10$ in hand;
losing $\frac{1}{3}$ of this, he had $\frac{2}{3}$ of it remaining, that is, $\frac{2}{3}(\frac{4}{5}x + 10)$;
and he then wins 3*s*, and so has $\frac{2}{3}(\frac{4}{5}x + 10) + 3$ shillings,
which, by the question, is equal to 3 guineas, or 63*s*;
hence $\frac{2}{3}(\frac{4}{5}x + 10) + 3 = 63$, whence $x = 100s = £5$.

Ex. 3. Find a number such that if $\frac{3}{8}$ of it be subtracted from 20, and $\frac{5}{11}$ of the remainder from $\frac{1}{4}$ of the original number, 12 times the second remainder shall be half the original number.

Let x = the number;
$\therefore 20 - \frac{3}{8}x$ = 1st remainder, and $\frac{1}{4}x - \frac{5}{11}(20 - \frac{3}{8}x)$=2nd remainder;
$\therefore 12\left\{\frac{1}{4}x - \frac{5}{11}(20 - \frac{3}{8}x)\right\} = \frac{1}{2}x$, by the question; whence $x = 24$.

Ex. 4. A certain number consists of two digits whose difference is 3; and, if the digits be inverted, the number so formed will be $\frac{4}{7}$ of the former: find the original number.

Let x = lesser digit, and $\therefore x + 3$ = the greater: then, since the value of a n° of two digits = ten-times the first digit + the second digit (thus $67 = 10 \times 6 + 7$), the n° in question $= 10(x + 3) + x$; similarly, the n° formed by the same digits inverted $= 10x + (x+3)$; hence, by question, $10x + (x + 3) = \frac{4}{7}\{10(x + 3) + x\}$, whence $x = 3$, $x + 3 = 6$, and the n° required is 63.

Ex. 5. A can do a piece of work in 10 days; but after he has been upon it 4 days, B is sent to help him, and they finish it together in 2 days. In what time would B have done the whole?

Let x = n° of days B would have taken, and W denote the work: $\therefore \frac{W}{10}, \frac{W}{x}$, are the portions of the work, which A, B would do in *one* day; hence in 4 days, A does $\frac{4W}{10}$, and in 2 days, A and B together do $\frac{2W}{10} + \frac{2W}{x}$: $\therefore \frac{4W}{10} + \frac{2W}{10} + \frac{2W}{x} = W$; whence $x = 5$.

It is plain that in the above, we might have omitted W altogether, or taken *unity* to represent the work, as follows: A, B do $\frac{1}{10}, \frac{1}{x}$ of the work respectively in *one* day, and therefore, reasoning just as before, $\frac{4}{10} + \frac{2}{10} + \frac{2}{x}$ = the whole work = 1.

[In all such questions the student should notice that, if a person does $\frac{m}{n}$ths of any work in 1 day, he will do $\frac{1}{n}$th of it in $\frac{1}{m}$th of a day, and therefore the *whole* work in $\frac{n}{m}$ days.

Thus if he does $\frac{3}{7}$ in one day, he will do $\frac{1}{7}$ in $\frac{1}{3}$ of a day, and $\therefore \frac{7}{7}$ or the *whole* in $\frac{7}{3} = 2\frac{1}{3}$ days].

Ex. 6. A cistern can be filled in half-an-hour by a pipe A, and emptied in 20′ by another pipe B: after A has been opened 20′, B is also opened for 12′, when A is closed, and B remains open for 5′ more, and now there are 13 gallons in the cistern: how much would it contain when full?

Let x = number of gallons that would fill the cistern: then, in 1′, A brings in $\frac{1}{30}x$ gals., and B carries out $\frac{1}{20}x$ gals.; but A is opened altogether for 32′, and B for 17′; $\therefore \frac{32}{30}x - \frac{17}{20}x = 13$, whence $x = 60$ gals.

Ex. 7. Find the time between two and three o'clock, at which the hour and minute-hand of a watch are exactly opposite each other.

Let x = number of minutes advanced by the *hour*-hand since two o'clock: then $12x$ = number of minutes advanced by the minute-hand, since it travels $60'$ while the other travels $5'$; but, by question, the minute-hand will have advanced $(10+x)+30 = x+40$ min.;

$\therefore 12x = x + 40$, whence $x = 3\frac{7}{11}$, and the time is 2h $43\frac{7}{11}'$.

Ex. 8. There are two bars of metal, the first containing 14 oz. of silver and 6 of tin, the second containing 8 of silver and 12 of tin; how much must be taken from each to form a bar of 20 oz. containing equal weights of silver and tin?

Let x = n° of oz. to be taken from first bar, $20 - x$ from second; now $\frac{14}{20}$ of the first bar, and therefore of *every* oz. of it, is silver;

and, similarly, $\frac{8}{20}$ of every oz. of the second bar is silver;

and there are to be altogether 10 oz. of silver in the compound;

$\therefore \frac{14}{20}x + \frac{8}{20}(20 - x) = 10$, whence $x = 6\frac{2}{3}$, and $20 - x = 13\frac{1}{3}$.

Ex. 44.

1. The stones which pave a square court would just cover a rectangular area, whose length is six yards longer, and breadth four yards shorter, than the side of the square: find the area of the court.

2. Out of a cask of wine, of which a fifth part had leaked away, 10 gallons were drawn, and then it was two-thirds full: how much did it hold?

3. A person brought a chaise, horse, and harness for £60; the horse cost twice as much as the harness, and the chaise half as much again as the horse and harness: what did he give for each?

4. The value of 50 coins, consisting of half-guineas and half-crowns, is £16 5*s*: how many are there of each?

5. *A*, after spending £10 less than a third of his yearly income, found that he had £45 more than half of it remaining: what was his income?

6. A boy, selling oranges, sells half his stock and one more to *A*, half of what remains and two more to *B*, and three that still remain to *C*: how many had he at first?

7. In a garrison of 2744 men, there are two cavalry soldiers to twenty-five infantry, and half as many artillery as cavalry: find the numbers of each.

8. A person dies worth £13,000: some of this he leaves to a Charity, and twelve times as much to his eldest son, whose share is half as much again as that of each of his two brothers, and two-thirds as much again as that of each of his five sisters: find the amount of the bequest to the Charity.

9. A farm of 270 acres is divided among A, B, C: A has 7 acres to 11 of B, and C has half as much again as A and B together: find the shares.

10. Divide 150 into two parts, such that if one be divided by 23 and the other by 27, the sum of the two quotients may be 6.

11. A had 18*s* in his purse, and B, when he had paid A two-thirds of his money, found that he had now remaining two-fifths of the sum which A now had: what had B at first?

12. The first digit of a certain number exceeds the second by 4, and when the number is divided by the sum of the digits, the quotient is 7: find it.

13. The length of a floor exceeds the breadth by 4 feet: if each had been increased by a foot, the area of the room would have been increased by 27 sq. ft.: find its original dimensions.

14. A met two beggars, B and C, and having a certain sum in his pocket, gave $\frac{3}{28}$ of it to B, and $\frac{3}{5}$ of the remainder to C: A had now 20*d* left; what had he at first?

15. In a mixture of copper, lead, and tin, the copper was 5 lb less than half the whole quantity, and the lead and tin each 5 lb more than a third of the remainder: find the respective quantities.

16. A sum of money was left for the poor widows of a parish, and it was found that, if each received 4*s* 6*d*, there would be 1*s* over; whereas, if each received 5*s*, there would be 10*s* short: how many widows were there? and what was the sum left?

17. A horse was sold at a loss for 40 guineas; but, if it had been sold for 50 guineas, the gain would have been three-fourths of the former loss: find its real value.

18. A can do a piece of work in 10 days, which B can do in 8: after A has been at work upon it 3 days, B comes to help him; in what time will they finish it?

19. There is a number of two digits, whose difference is 2, and

if it be diminished by half as much again as the sum of the digits, the digits will be inverted: find it.

20. *A* and *B* have the same income: *A* lays by a fifth of his: but *B*, by spending annually £80 more than *A*, at the end of 4 years finds himself £220 in debt. What was their income?

21. A number of troops being formed into a solid square, it was found there were 60 over; but, when formed into a column with 5 men more in front than before and 3 less in depth, there was just one man wanting to complete it. Find the number.

22. A person has travelled altogether 3036 miles, of which he has gone seven miles by water to four on foot, and five by water to two on horseback: how many did he travel each way?

23. *A* and *B* can reap a field together in 7 days, which *A* alone could reap in 10 days: in what time could *B* alone reap it?

24. A cistern can be filled in 15′ by two pipes, *A* and *B*, running together: after *A* has been running by itself for 5′, *B* is also turned on, and the cistern is filled in 13′ more: in what time would it be filled by each pipe separately?

25. What is the first hour after 6 o'clock, at which the two hands of a watch are (i) directly opposite, and (ii) at right angles, to each other?

26. A person played twenty games at chess for a wager of 3*s* to 2*s*, and upon the whole he gained 5*s*: how many games did he win?

27. I wish to enclose a piece of ground with palisades; and find that, if I set them a foot asunder, I shall have too few by 150, whereas, if I set them a yard asunder, I shall have too many by 70: what is the circuit of the piece of ground?

28. *A* and *B* began to pay their debts: *A*'s money was at first $\frac{2}{3}$ of *B*'s; but after *A* had paid £1 less than $\frac{2}{3}$ of his money, and *B* had paid £1 more than $\frac{7}{8}$ of his, it was found that *B* had only half as much as *A* had left. What sum had each at first?

29. *A* can build a wall in 8 days, which *A* and *B* can do together in 5 days: how long would *B* take to do it alone? and how long after *B* has begun should *A* begin, so that, finishing it together, they may each have built half the wall?

30. A person wishing to sell a watch by lottery, charges 6*s* each for the tickets, by which he gains £4; whereas, if he had made a third as many tickets again and charged 5*s* each, he would have gained as many shillings as he had sold tickets: what was the value of the watch?

31. A mass of copper and tin weighs 80 lbs, and for every 7 lbs of copper there are 3 lbs of tin: how much copper must be added to the mass, that for every 11 lbs of copper there may by 4 lbs of tin?

32. *A* does $\frac{5}{9}$ of a piece of work in 10 days, when *B* comes to help him, and they take three days more to finish it: in what time would they have done the whole, each separately, or both together?

33. A cistern can be filled by two pipes, *A* and *B*, in 24′ and 30′ respectively, and emptied by a third *C* in 20′: in what time would it be filled, if all three were running together?

34. *A* and *B* were employed together for 50 days, each at 5*s* a day, during which time *A*, by spending 6*d* a day less than *B*, had saved three times as much as *B*, and $2\frac{1}{2}$ days' pay besides; what did each spend per day?

35. Divide £149 among *A*, *B*, *C*, *D*, so that *A* may have half as much again as *B*, and a third as much again as *B* and *C* together: and *D* a fourth as much again as *A* and *C* together.

36. There are two silver cups and one cover for both. The first weighs 12 oz, and, with the cover, weighs twice as much as the other cup without it; but the second with the cover weighs a third as much again as the first without it. Find the weight of the the cover.

37. A man could reap a field by himself in 20 hrs, but, with his son's help for 6 hrs, he could do it in 16 hrs: how long would the son be in reaping the field by himself?

38. A horsekeeper, not having room in his stables for 8 of his horses, built so as to increase his accommodation by one half, and now has room for 8 more than his whole number: how many horses had he?

39. A grocer bought tea at 6*s* 6*d* per lb, and a third as many lbs again of coffee at 2*s* 6*d* per lb; he sold the tea at 8*s*, and the coffee at 2*s* 3*d*, and so gained 5 guineas by the bargain: how many lbs of each did he buy?

40. Find a number of three digits, each greater by unity than that which follows it, so that its excess above one-fourth of the number formed by inverting the digits shall be 36 times the sum of the digits.

41. A man and his wife could drink a cask of beer in 20 days, the man drinking half as much again as his wife; but, $\frac{13}{23}$ of a gallon having leaked away, they found that it only lasted them

together for 18 days, and the wife herself for two days longer: how much did it contain when full?

42. A and B have each a sum of money given them, which will support their families for 10 and 12 days respectively; but A's money would support B's family for 15 days, and B's money would support A's family for 7 days, with 2*s* 6*d* over: what were the sums?

43. A person being asked how many ducks and geese he had in his yard, said, If I had 8 more of each, I should have 8 ducks for 7 geese, and if I had 8 less of each, I should have 7 ducks for 6 geese: how many had he of each?

44. A man, woman, and child could reap a field in 30 hrs, the man doing half as much again as the woman, and the woman two-thirds as much again as the child: how many hours would they each take to do it separately?

45. If 19 lbs of gold weigh 18 lbs in water, and 10 lbs of silver weigh 9 lbs in water, find the quantity of gold and silver in a mass of gold and silver, weighing 106 lbs in air and 99 lbs in water.

46. From each of a number of foreign gold coins a person filed a fifth part, and had passed two-thirds of them, gaining thereby 35*s*, when the rest were seized as light coin, except one with which the man decamped, having lost upon the whole half as much as he had gained before: how many coins were there at first?

47. A and B start to run a race: at the end of $5'$, when A has run 900 yards and has outstripped B by 75 yards, he falls; but, though he loses ground by the accident, and for the rest of the course makes 20 yards a minute less than before, he comes in only half-a-minute behind B. How long did the race last?

48. A and B can reap a field together in 12 hrs, A and C in 16 hrs, and A by himself in 20 hrs: in what time could (i) B and C together, (ii) A, B, and C, together, reap it?

49. Fifteen guineas should weigh 4 oz: but a parcel of light gold, having been weighed and counted, was found to contain 9 more guineas than was supposed from the weight, and it appeared that 21 of these coins weighed the same as 20 true guineas: how many were there altogether?

50. A, B, C travel from the same place at the rate of 4, 5, and 6 miles an hour respectively, and B starts two hours after A: how long after B must C start, in order that they may both overtake A at the same moment?

Simultaneous Equations of one Dimension.

94. If *one* equation contain *two* unknown quantities, there are an infinite number of pairs of values of these by which it may be satisfied.

Thus in $x = 10 - 2y$, if we give *any value* to y, we shall get a corresponding value for x, by which pair of values the equation will of course be satisfied; if, for example, we take $y = 1$, we shall get $x = 10 - 2 = 8$; if $y = 2$, $x = 6$; if $y = 3$, $x = 4$, &c.

One equation then between *two* unknown quantities admits of an infinite number of solutions; but if we have as many different equations, as there are quantities, the number of solutions will be limited.

Thus, while each of the equatious $x = 10 - 2y$, $4x + 4 = 3y$, separately considered, is satisfied by an infinite number of pairs of values of x and y, there will only be found *one* pair common to both, viz. $x = 2$, $y = 4$, which are therefore the roots of the pair of equations, $x = 10 - 2y$, and $4x + 4 = 3y$.

Equations of this kind, which are to be satisfied by the *same* pair or pairs of values of x and y, are called *simultaneous equations.*

If there be *three* unknowns, there must be *three* equations, and so on: and, moreover, these equations must all be *different* from one another; *i. e.* must *all* express *different* relations between the unknown quantities.

Thus, if we had the equation $x = 10 - 2y$, it would be of no use to join with it the equation $2x = 20 - 4y$ (which is obtained by merely doubling it), or any other, derived like this, immediately from the former; since this expresses no new relation between x and y, but repeats in another form the same as before.

It may be observed, that if any two or more equations be given, any equations formed by adding or subtracting any multiples of these equations, will be also *true*, though expressing, in reality, no *new* relations between the quantities.

Thus if $x + 3y + 4z = 9$, and $3x - 2y + 17z = 25$; then, subtracting the second from three times the first, we have $11y - 5z = 2$.

95. There are generally given three methods for solving simultaneous equations of two unknowns; but the object aimed at is the same in each, viz. to combine the two equations in such a manner as to expel, or, as the phrase is, *eliminate* from the result one quantity, and so get an equation of *one* unknown only.

96. *First method.*—Multiply, when possible, one equation by some number, that may make the coeff. of x or y in it the same as in the other; then, adding or subtracting the two equations, according as these equal quantities have different or same signs, these terms will destroy each other, and the elimination will be effected.

Ex. 1.
$$\left.\begin{aligned} 4x + y &= 34 \\ 4y + x &= 16 \end{aligned}\right\} \quad \begin{aligned} &\text{(i)} \\ &\text{(ii)} \end{aligned}$$

Here, mult. (ii) by 4, $16y + 4x = 64$,

but $y + 4y = 34$; (i)

$\therefore$ subtracting, $15y = 30$, and $\therefore y = 2$;

and (ii) $x = 16 - 4y = 16 - 8 = 8$.

Ex. 2.
$$\left.\begin{aligned} 4x - y &= 7 \\ 3x + 4y &= 29 \end{aligned}\right\} \quad \begin{aligned} &\text{(i)} \\ &\text{(ii)} \end{aligned}$$

Here $3x + 4y = 29$,

and, mult. (i) by 4, $16x - 4y = 28$;

$\therefore$ adding, $19x = 57$, and $\therefore x = 3$;

and (i) $y = 4x - 7 = 12 - 7 = 5$.

Sometimes we cannot make the coefficients equal by multiplying only one of the equations; but shall have to multiply both by some numbers, which it will be easy to perceive in any case.

Ex. 3.
$$\left.\begin{aligned} 2x + 3y &= 4 \\ 3x - 2y &= -7 \end{aligned}\right\} \quad \begin{aligned} &\text{(i)} \\ &\text{(ii)} \end{aligned}$$

Mult. (i) by 3, $6x + 9y = 12$

— (ii) by 2, $6x - 4y = -14$

subtracting, $13y = 26$, and $\therefore y = 2$;

and (i) $2x = 4 - 3y = 4 - 6 = -2$; $\therefore x = -1$.

97. *Second method.*—Express one of the unknown quantities in terms of the other by means of one of the equations, and put this value for it in the other equation.

Ex. 4. $\left.\begin{array}{r} 7x + \frac{1}{5}(2y + 4) = 16 \\ 3y - \frac{1}{4}(x + 2) = 8 \end{array}\right\}$ or reducing, $\left.\begin{array}{r} 35x + 2x = 76 \\ 12y - x = 34 \end{array}\right\}$ $\begin{array}{c} \text{(i)} \\ \text{(ii)} \end{array}$

Here from (ii) $x - 12y - 34$, and from (i) $35(12y - 34) + 2y = 76$,

whence $y = 3$, and $\therefore x = 2$.

98. *Third method.*—Express the *same* quantity in terms of the other in both equations, and put these values equal.

Ex. 5. $\left.\begin{array}{r} 5x - \frac{1}{4}(5y + 2) = 32 \\ 3y + \frac{1}{3}(x + 2) = 9 \end{array}\right\}$ or reducing, $\left.\begin{array}{r} 20x - 5y = 130 \\ 9y + x = 25 \end{array}\right\}$ $\begin{array}{c} \text{(i)} \\ \text{(ii)} \end{array}$

Here in (i), $y = \frac{1}{5}(20x - 130)$, in (ii $y = \frac{1}{9}(25 - x)$;

$\therefore \frac{1}{5}(20x - 130) = \frac{1}{9}(25 - x)$, whence $x = 7$, $y = 2$.

The first of these methods is generally to be preferred; but the second may be used with advantage, whenever either x or y has a coefficient *unity* in one of the equations.

Ex. 45.

1. $\left.\begin{array}{r} 2x + 9y = 11 \\ 4x + y = 5 \end{array}\right\}$

2. $\left.\begin{array}{r} x + y = a \\ ax + by = b^2 \end{array}\right\}$

3. $\left.\begin{array}{r} 2x - y = 8 \\ 2y + x = 9 \end{array}\right\}$

4. $\left.\begin{array}{r} ax + y = b \\ x + by = a \end{array}\right\}$

5. $\left.\begin{array}{r} 2x - 9y = 11 \\ 3x - 12y = 15 \end{array}\right\}$

6. $\left.\begin{array}{r} bx + ay = b \\ ax - by = a \end{array}\right\}$

7. $\left.\begin{array}{r} 2x + 3y - 8 = 0 \\ 7x - y - 5 = 0 \end{array}\right\}$

8. $\left.\begin{array}{r} ax = by \\ x + y = c \end{array}\right\}$

9. $\left.\begin{array}{r} 5x + 4y = 58 \\ 3x + 7y = 67 \end{array}\right\}$

10. $\left.\begin{array}{r} x(y+7) = y(x+1) \\ 2x + 20 = 3y + 1 \end{array}\right\}$

11. $\left.\begin{array}{r} \frac{1}{2}x + \frac{1}{3}y = 13 \\ \frac{1}{5}x + \frac{1}{8}y = 5 \end{array}\right\}$

12. $\left.\begin{array}{r} \frac{1}{9}x + \frac{1}{8}y = 43 \\ \frac{1}{8}x + \frac{1}{9}y = 42 \end{array}\right\}$

13. $\left.\begin{array}{r} \frac{x}{a} - \frac{y}{b} = m \\ \frac{x}{c} + \frac{y}{d} = n \end{array}\right\}$

14. $\left.\begin{array}{r} 2x - \frac{y-3}{5} = 4 \\ 3y + \frac{x-2}{3} = 9 \end{array}\right\}$

15. $\left.\begin{array}{r} \frac{x}{b} + \frac{y}{c} = 1 \\ \frac{ax}{c} + \frac{by}{a} = 0 \end{array}\right\}$

16. $\left.\begin{array}{r} ax + by = c^2 \\ \frac{a}{b+y} - \frac{b}{a+x} = 0 \end{array}\right\}$

17. $\left.\begin{array}{r} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{x}{b} - \frac{y}{a} = 1 \end{array}\right\}$

18. $\left.\begin{array}{r} \frac{x}{a} + \frac{y}{b} = 1 - \frac{x}{c} \\ \frac{y}{a} - \frac{x}{b} = 1 + \frac{y}{c} \end{array}\right\}$

19. $\left.\begin{array}{r} \frac{1}{6}(2x + 3y) + \frac{1}{3}x = 8 \\ \frac{1}{2}(7y - 3x) - y = 11 \end{array}\right\}$

20. $\left.\begin{array}{r} \frac{1}{4}(2x - y) + 1 = \frac{1}{5}(7 + x) \\ \frac{1}{6}(3 - 4x) + 3 = \frac{1}{3}(5y - 7) \end{array}\right\}$

21. $\left.\begin{aligned} x - \tfrac{1}{7}(y-2) &= 5 \\ 4y - \tfrac{1}{3}(x+10) &= 3 \end{aligned}\right\}$

22. $\left.\begin{aligned} \tfrac{106}{63}y + \tfrac{1}{7}(x-6y+1) &= \tfrac{1}{9}(x-3) \\ \tfrac{1}{9}(x-5y+8) &= \tfrac{1}{7}(3x-13y) + \tfrac{55}{63} \end{aligned}\right\}$

23. $\left.\begin{aligned} \tfrac{1}{10}(3x+4y+3) - \tfrac{1}{15}(3x-y) &= 5 + \tfrac{1}{5}(y-8) \\ \tfrac{1}{12}(9y+5x-8) - \tfrac{1}{4}(x+y) &= \tfrac{1}{11}(7x+6) \end{aligned}\right\}$

24. $\left.\begin{aligned} 2x - \frac{y+3}{4} &= 7 + \frac{3y-2x}{5} \\ 4y - \frac{8-x}{3} &= 24\tfrac{1}{2} - \frac{2y+1}{2} \end{aligned}\right\}$

25. $\left.\begin{aligned} x - \frac{2y-x}{23-x} &= 20 - \frac{59-2x}{2} \\ y + \frac{y-3}{x-18} &= 30 - \frac{73-3y}{3} \end{aligned}\right\}$

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99. Simultaneous equations of three unknown quantities are solved by eliminating one of them by means of any pair of the equations, and then the *same* one by means of another pair: we shall thus have two equations involving the same two unknown quantities, which may now be solved by the preceding rules.

Similarly for those of more than three unknowns.

Ex 1. $\left.\begin{aligned} x - 2y + 3z &= 2 \\ 2x - 3y + z &= 1 \\ 3x - y + 2z &= 9 \end{aligned}\right\}$ $\begin{aligned} &\text{(i)} \\ &\text{(ii)} \\ &\text{(iii)} \end{aligned}$

$$\begin{aligned} \text{From (i)}\quad 2x - 4y + 6z &= 4 \\ \text{(ii)}\quad 2x - 3y + z &= 1 \\ \hline \therefore\quad -y + 5z &= 3 \quad (\alpha) \end{aligned}$$

$$\begin{aligned} \text{Again (i)}\quad 3x - 6y + 9z &= 6 \\ \text{(iii)}\quad 3x - y + 2z &= 9 \\ \hline \therefore\quad -5y + 7z &= -3\ (\beta) \\ \text{but}\quad -5y + 25z &= 15\ (\alpha) \\ \hline \therefore\quad -18z &= -18, \text{ and } z = 1: \end{aligned}$$

hence (α) $y = 5z - 3 = 2$, and (i) $x = 2 + 2y - 3z = 3$.

Ex. 2. $\left.\begin{aligned} \frac{a}{x} + \frac{b}{y} &= \frac{1}{r} \\ \frac{a}{x} + \frac{c}{z} &= \frac{1}{q} \\ \frac{b}{y} + \frac{c}{z} &= \frac{1}{p} \end{aligned}\right\}$ $\begin{aligned} &\text{(i)} \\ &\text{(ii)} \\ &\text{(iii)} \end{aligned}$

From (ii) and (iii) $\dfrac{a}{x} - \dfrac{b}{y} = \dfrac{1}{q} - \dfrac{1}{p}$

and (i) $\dfrac{a}{x} + \dfrac{b}{y} = \dfrac{1}{r}$

$$\therefore \frac{2a}{x} = \frac{1}{q} + \frac{1}{r} - \frac{1}{p} = \frac{(q+r)p - qr}{pqr},$$

and $x = \dfrac{2pqra}{(q+r)p - qr}$; so $y = \dfrac{2pqrb}{(p+r)q - pr}$, $z = \dfrac{2pqrc}{(p+q)r - pq}$,

which latter values may be written down at once from the *Symmetry* of the equations, since it is obvious that the values of y and z will be of the same *form* as that of x, only interchanging (for y) a with b, and p with q, and (for z) a with c, and p with r.

Ex. 46.

1. $\left.\begin{array}{l}2x+3y+4z=20\\3x+4y+5z=26\\3x+5y+6z=31\end{array}\right\}$

2. $\left.\begin{array}{l}5x+3y=65\\2y-z=11\\3x+4z=57\end{array}\right\}$

3. $\left.\begin{array}{l}3x+2y-z=20\\2x+3y+6z=70\\x-y+6z=41\end{array}\right\}$

4. $\left.\begin{array}{l}x+y+z=5\\x+y=z-7\\x-3=y+z\end{array}\right\}$

5. $\left.\begin{array}{l}x+2y=7\\y+2z=2\\3x+2y=z-1\end{array}\right\}$

6. $\left.\begin{array}{l}a=y+z\\b=x+z\\c=x+y\end{array}\right\}$

7. $\left.\begin{array}{l}xy=x+y\\xz=2(x+z)\\yz=3(y-z)\end{array}\right\}$

8. $\left.\begin{array}{l}2(x-y)=3z-2\\x+1=3(y+z)\\2x+3z=4(1-y)\end{array}\right\}$

9. $\left.\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=12-\frac{1}{6}z\\\frac{1}{2}y+\frac{1}{3}z=8+\frac{1}{6}x\\\frac{1}{2}x+\frac{1}{3}z=10\end{array}\right\}$

10. $\left.\begin{array}{l}y+\frac{1}{3}z=\frac{1}{5}x+5\\\frac{1}{4}(x-1)-\frac{1}{5}(y-2)=\frac{1}{10}(z+3)\\x-\frac{1}{3}(2y-5)=1\frac{3}{4}-\frac{1}{12}z\end{array}\right\}$

Ex. 47.

1. What fraction is that, to the numerator of which if 7 be added, its value is $\frac{2}{3}$; but if 7 be taken from the denominator its value is $\frac{3}{8}$?

2. A bill of 25 guineas was paid with crowns and half guineas; and twice the number of half guineas exceeded three times that of the crowns by 17: how many were there of each?

3. *A* and *B* received £5 17*s* for their wages, *A* having been employed 15, and *B* 14 days; and *A* received for working four days 11*s* more than *B* did for three days: what were their daily wages?

4. A farmer parting with his stock sells to one person 9 horses and 7 cows for £300; and to another, at the same prices, 6 horses and 13 cows for the same sum: what was the price of each?

5. A draper bought two pieces of cloth for £12 13*s*, one being 8*s* and the other 9*s* per yard. He sold them each at an advanced price of 2*s* per yard, and gained by the whole £3. What were the lengths of the pieces?

6. There is a number of two digits, which, when divided by their sum, gives the quotient 4; but if the digits be inverted, and the number thus formed be increased by 12, and then divided by their sum, the quotient is 8. Find the number.

7. A rectangular bowling-green having been measured, it was observed that, if it were 5 feet broader and 4 feet longer, it would contain 116 feet more; but, if it were 4 feet broader and 5 feet longer, it would contain 113 feet more. Find its present area.

8. Find three numbers *A*, *B*, *C*, such that *A* with half of *B*, *B* with a third of *C*, and *C* with a fourth of *A*, may each be 1000.

9. A train left Cambridge for London with a certain number of passengers, 40 more second-class than first-class; and 7 of the former would pay together $2s$ less than 4 of the latter. The fare of the whole was £55. But they took up, half-way, 35 more second-class and 5 first-class passengers, and the whole fare now received was $\frac{1}{5}$ as much again as before. What was the first-class fare, and the whole number of passengers at first?

10. A person rows from Cambridge to Ely, a distance of 20 miles, and back again, in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

11. The sum of the two digits of a certain number is six times their difference, and the number itself exceeds six times their sum by 3: find it.

12. A grocer bought tea at $10s$ per lb, and coffee at $2s$ $6d$ per lb, to the amount altogether of £31 $5s$: he sold the tea at $8s$, and the coffee at $4s$ $6d$, and gained £5 by the bargain: how many lbs of each did he buy?

13. A and B can do a piece of work together in 12 days, which B working for 15 days and C for 30 would together complete: in 10 days they would finish it, working all three together; in what time could they separately do it?

14. A sum of £12 $18s$ might be distributed to the poor of a parish by giving $\frac{1}{2}$ a crown to each man and $1s$ to each woman and each child, or $\frac{1}{3}$ a crown to each woman and $1s$ to each man and each child, or $\frac{1}{4}$ a crown to each child and $1s$ to each man and each woman: how many were there in all?

15. Divide the numbers 80 and 90 each into two parts, so that the sum of one out of each pair may be 100, and the difference of the others 30.

16. Some smugglers found a cave, which would just exactly hold the cargo of their boat, viz. 13 bales of silk and 33 casks of rum. While unloading, a revenue cutter came in sight, and they were obliged to sail away, having landed only 9 casks and 5 bales, and filled one-third of the cave. How many bales separately, or how many casks, would it hold?

17. A person spends $2s$ $6d$ in apples and pears, buying the apples at four, and the pears at five a penny; and afterwards

accommodates a neighbour with half his apples and a third of his pears for 13*d*. How many of each did he buy?

18. A party was composed of a certain number of men and women, and, when four of the women were gone, it was observed that there were left just half as many men again as women: they came back, however, with their husbands, and now there were only a third as many men again as women. What were the original numbers of each?

19. A and B play at bowls, and A bets B 3*s* to 2*s* on every game: after a certain number of games, it appears that A has won 3*s*; but had he ventured to bet 5*s* to 2*s*, and lost one game more out of the same number, he would have lost 30*s*. How many games did they play?

20. A person, being asked how many oranges he had bought, said, "These cost me 1*s* 6*d* per dozen; but if I had got the five into the bargain which I asked for, they would have cost me $2\frac{1}{2}d$ a dozen less." How many had he?

21. Having 45*s* to give away among a certain number of persons, I find that if I give 3*s* to each man and 1*s* to each woman, I shall have 1*s* too little, but that, by giving 2*s* 6*d* to each man and 1*s* 6*d* to each woman, I may distribute the sum exactly. How many were there of men and women?

22. Find the number of three digits, the last two alike, such that the number formed by the digits inverted may exceed twice the original number by 42, and also the number formed by putting the *single* figure in the midst by 27.

23. A party at a tavern, having to pay their reckoning, and being a third as many men again as women, agree that each man shall pay half as much again as each woman; but, a man and his wife having gone off without paying their share, 10*d*, the rest had each to pay 2*d* more. What was the reckoning?

24. A, B, C, sit down to play: in the first game, A loses to each of B and C as much as each of them has, in the second B loses similarly to each of A and C, and in the third C loses similarly to each of A and B; and now they have each 24*s*. What had they each at first?

CHAPTER VIII.

INDICES, AND SURDS.

100. It was stated in (45), that, when any root of a quantity cannot be exactly obtained, it is expressed by the use of the sign of evolution, as $\sqrt{3}$, $\sqrt[3]{2ac}$, $\sqrt[5]{a^2+c^2}$, and called an *Irrational* or *Surd* quantity.

It was also stated in (46) that there cannot be any *even* root of a *negative* quantity; but that such roots may be expressed in the form of surds, as $\sqrt{-3}$, $\sqrt[4]{-a^2}$, $\sqrt[6]{-(a^2+b^2)}$, and are then called *impossible* or *imaginary* quantities.

These we shall consider more at length in this chapter.

It was seen in (20), that powers of the same quantity were multiplied by adding their indices; we shall now prove this rule to be *generally* true, which was there only shewn to be true in particular instances.

101. *To prove that* $a^m \times a^n = a^{m+n}$, *when* m *and* n *are any positive integers.*

Since by (9) $a^m = a \times a \times$ &c. (m factors)
and $a^n = a \times a \times$ &c. (n factors),

it follows that

$a^m \times a^n = a \times a \times$ &c. (m factors) $\times a \times a \times$ &c. (n factors)
$= a \times a \times$ &c. ($m+n$ factors) $= a^{m+n}$, by (9).

102. Hence $(a^m)^n = a^{mn} = (a^n)^m$;

for $(a^m)^n = a^m . a^m . a^m .$&c. n factors $= a^{m+m+m+\text{\&c. } n \text{ terms}} = a^{mn}$,
and $(a^n)^m = a^n . a^n . a^n .$&c. m factors $= a^{n+n+n+\text{\&c. } m \text{ terms}} = a^{nm}$;

$\therefore$ since $a^{mn} = a^{nm}$, we have $(a^m)^n = a^{mn} = (a^n)^m$:

that is, *the* n^{th} *power of the* m^{th} *power of* a $=$ *the* m^{th} *power of the* n^{th} *power of* a, and either of them is found by multiplying the two indices.

103. Hence also $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$:

for let $\sqrt[n]{a^m} = x^m$, then $a^m = (x^m)^n = (x^n)^m$ by (102);

hence $a = x^n$, and $\therefore \sqrt[n]{a} = x$, and $(\sqrt[n]{a})^m = x^m$;

but also, by our first assumption, $\sqrt[n]{a^m}=x^m$;

hence we have $\sqrt[n]{a^m}=(\sqrt[n]{a})^m$;

that is, *the* n[th] *root of the* m[th] *power of* a = *the* m[th] *power of the* n[th] *root of* a.

104. These results refer as yet only to positive integral indices, which (9) were first used to express briefly the repetition of the same factor in any product.

But now, suppose we write down a quantity, with a positive *fraction* for an index, such as $a^{\frac{p}{q}}$, and agree that such a symbol shall be treated by the same law of Multiplication as if the index were an *integer*, viz. $a^m.a^n=a^{m+n}$:—what would such a symbol, so treated, denote?

Since it follows from this law, in the case of positive *integers*, that $(a^m)^n=a^{mn}$, we should have here also $(a^{\frac{p}{q}})^q=a^{\frac{pq}{q}}=a^p$; and hence it appears that $a^{\frac{p}{q}}$ would denote such a quantity as, *when raised to the* q[th] *power*, becomes equal to a^p. But that quantity, whose q[th] power$=a^p$, is (10), the q[th] *root of* a^p; and, therefore, $a^{\frac{p}{q}}=\sqrt[q]{a^p}$, or$=(\sqrt[q]{a})^p$ by (103).

Hence, when a fractional index is employed with any quantity, the *numerator* denotes a *power*, and the *denominator* a *root* to be taken of it.

Thus $a^{\frac{1}{2}}$ = 2[nd] root of 1[st] power of $a=\sqrt{a}$, $a^{\frac{1}{3}}=\sqrt[3]{a}$, $a^{\frac{1}{4}}=\sqrt[4]{a}$, &c.

$a^{\frac{2}{3}}$ = *cube* root of *square* of $a=\sqrt[3]{a^2}$,

or = *square* of *cube* root of $a=(\sqrt[3]{a})^2$;

so $a^{\frac{3}{4}}=\sqrt[4]{a^3}$ or $(\sqrt[4]{a})^3$; $a^{\frac{1}{2}}=a^{\frac{2}{4}}=a^{\frac{3}{6}}=$&c., or $\sqrt{a}=\sqrt[4]{a^2}=\sqrt[6]{a^3}=$&c.

105. Again, if we write down a quantity with a *negative* index, as a^{-p} (where p may now be integral or fractional), and agree that this symbol shall be treated by the same law of Mult[n] as if the index were positive, what would such a symbol, so treated, denote?

By this law we should have $a^{m+p} \times a^{-p} = a^{m+p-p} = a^m$;

but we have also $$a^{m+p} \div a^p = \frac{a^{m+p}}{a^p} = \frac{a^m . a^p}{a^p} = a^m:$$

so that, to *multiply* by a^{-p}, is the same as to *divide* by a^p: and, therefore, $1 \times a^{-p} = 1 \div a^p$, or $a^{-p} = \frac{1}{a^p}$.

Hence, any quantity with a *negative* index denotes the *reciprocal* of the same with the same *positive* index.

Thus $a^{-1} = \frac{1}{a}$, $a^{-3} = \frac{1}{a^3}$, $a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}$, or $= \sqrt{a^{-1}} = \sqrt{\frac{1}{a}}$;

$$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}} \quad \text{or} = \sqrt[3]{a^{-2}} = \sqrt[3]{\frac{1}{a^2}}.$$

Hence also any power in the numerator of a quantity may be removed into the denominator, and *vice versâ*, by merely changing the sign of its index.

$$\text{Thus } a^{-3}b^2c^{-1} = \frac{a^{-3}b^2}{c} = \frac{a^{-3}}{b^{-2}c} = \frac{b^2c^{-1}}{a^3} = \&c.$$

106. Lastly, if we write down a quantity with *zero* for an index, as a^0, and agree that this symbol shall be treated as if the index were an actual number,—what then would it denote?

Since, by this law, $a^0 \times a^m = a^{0+m} = a^m$, it follows that a^0 is only equivalent to 1, whatever be the value of a.

In actual practice, such a quantity as a^0 would only occur in certain cases, where we wish to keep in mind from what a certain number may have arisen: thus $(a^3 + 2a^2 + 3a + \&c) \div a^2 = a + 2 + 3a^{-1} + \&c.$, where the 2 has lost all sign of its having been originally a coeff. of some power of a; if, however, we write the quotient $a + 2a^0 + 3a^{-1} + \&c.$, we preserve an indication of this, and have, as it were, a connecting link between the positive and negative powers of a.

The quantity $a^{\frac{p}{q}}$ is still called *a to the power of* $\frac{p}{q}$, and similarly in the case of a^{-m}, a^0; but the word *power* has here lost its original meaning, and denotes merely *a quantity with an index*, whatever that index may be, subject, in all cases, to the Law, $a^m . a^n = a^{m+n}$.

Ex. 48.

Express, with *fractional* indices,

1. $\sqrt{x^3} + \sqrt[3]{x^4} + (\sqrt{x})^5 + (\sqrt[3]{x})^2$; $\sqrt[3]{a^3b^2} + \sqrt[8]{a^2b^4} + \sqrt[3]{ab^6} + \sqrt[3]{a^5b^3}$.
2. $a\sqrt[3]{b^8} + (\sqrt{a})^5 + \sqrt[3]{a^6b} + \sqrt[6]{a^6b^4}$; $\sqrt[4]{a^2b^8} + a(\sqrt[3]{b})^6 + \sqrt[5]{a^2b^{10}} + \sqrt{a^3b^4}$.

Express, with *negative* indices, so as to remove all powers,

(i) into the numerators, and (ii) into the denominators,

3. $\frac{1}{a} + \frac{2}{b^2} + \frac{3}{c^3} + \frac{4a}{b} + \frac{5b}{a}$; $\frac{a^3}{b^3} + \frac{3a^2}{b} + \frac{5a}{b^2} + \frac{4b}{a^2} + \frac{2b^2}{a^3}$.
4. $\frac{a^3}{3b^2c^2} + \frac{4c^2}{a^2b} + \frac{2bc}{a} + \frac{1}{3abc}$; $\frac{ab}{2\sqrt[3]{c}} + \frac{2b^2c^2}{3\sqrt{a^3}} + \frac{3}{4\sqrt[3]{a^2bc^2}} + \frac{5c}{a\sqrt[4]{b^3}}$.

Express, with the *sign of Evolution*,

5. $a^{\frac{1}{2}} + 2a^{\frac{2}{3}} + 3a^{\frac{3}{4}} + 4a^{\frac{1}{5}} + a^{\frac{3}{4}}$; $\frac{a^{\frac{1}{4}}}{b^{\frac{3}{4}}} + \frac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{2c^{\frac{1}{2}}} + \frac{2a^{\frac{1}{4}}c^{\frac{3}{4}}}{3b^{\frac{3}{2}}} + \frac{b^{\frac{2}{5}}c^{\frac{3}{5}}}{4a^{\frac{1}{5}}} + \frac{b^{\frac{1}{6}}c^{\frac{1}{6}}}{5a^{\frac{3}{4}}}$.

Express, with *positive* indices, and with the sign of Evolution,

6. $a^{-1}bc + ab^{-2}c + a^{-1}b^{-1}c^{-1} + a^{-1}b^{-2}c^3$; $a^{-\frac{2}{5}} + a^{\frac{1}{2}}b^{-\frac{4}{3}} + a^{-\frac{3}{2}}b^{\frac{2}{3}} + b^{-\frac{5}{3}}$.
7. $\frac{a^{-2}b^{-2}}{c^{-1}} + \frac{2a}{b^{-1}c^{-1}} + \frac{3b^{-1}c^{-2}}{a^{-3}} + \frac{1}{a^{-1}b^{-2}c^{-3}}$, $\frac{a^{-2}}{b^{-\frac{1}{3}}} + \frac{b^{-\frac{2}{3}}}{a^{-\frac{3}{2}}} + \frac{b^{-\frac{3}{4}}}{a^{-\frac{2}{3}}} + \frac{a^{-\frac{1}{2}}}{b^{-2}}$.

107. It follows, then, that, *whatever* be the indices,

$$a^m \times a^n = a^{m+n}, \ \frac{a^m}{a^n} = a^m \times \frac{1}{a^n} = a^m \times a^{-n} = a^{m-n}, \ (a^m)^n = a^{mn};$$

so that (i) to *multiply* any powers of the same quantity, we must *add* the indices, (ii) to *divide* any one power of a quantity by another, we must *subtract* the index of the divisor from that of the dividend, and (iii) to obtain any *power of a power* of a quantity, we must multiply together the two indices.

Thus $a^3 \times a^2 = a^{3-2} = a$, $a^3 \div a^{-\frac{1}{2}} = a^{3+\frac{1}{2}} = a^{\frac{7}{2}}$, $a^{-\frac{1}{2}} \div a^{-\frac{3}{5}} = a^{-\frac{1}{2}+\frac{3}{5}} = a^{\frac{1}{10}}$,

$$(a^3)^{-2} = a^{-6}, \ (a^{-3})^{-\frac{1}{2}} = a^{\frac{3}{2}}, \ \{(a^{-\frac{1}{2}})^{\frac{2}{3}}\}^{-12} = a^4,$$

$$\left\{\sqrt{ab^{-2}\sqrt{ab}}\right\}^4 = \left\{a^{\frac{1}{2}}b^{-1} \,.\, a^{\frac{1}{4}}b^{-\frac{1}{4}}\right\}^4 = (a^{\frac{3}{4}}b^{-\frac{3}{4}})^4 = a^3b^{-3}.$$

Ex. 1. *Multiplication.*

$$\begin{array}{l} a^{\frac{5}{2}} + a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} + ab + a^{\frac{1}{2}} b^{\frac{4}{3}} + b^{\frac{5}{3}} \\ a^{\frac{1}{2}} - b^{\frac{1}{3}} \\ \hline a^3 + a^{\frac{5}{2}} b^{\frac{1}{3}} + a^2 b^{\frac{2}{3}} + a^{\frac{3}{2}} b + a b^{\frac{4}{3}} + a^{\frac{1}{2}} b^{\frac{5}{3}} \\ \quad - a^{\frac{5}{2}} b^{\frac{1}{3}} - a^2 b^{\frac{2}{3}} - a^{\frac{3}{2}} b - a b^{\frac{4}{3}} - a^{\frac{1}{2}} b^{\frac{5}{3}} - b^2 \\ \hline a^3 \quad * \quad * \quad * \quad * \quad * \quad - b^2 \end{array}$$

Ex. 2. *Division.*

$$x^{\frac{3}{2}} - 4ax^{\frac{1}{2}} + 2a^{\frac{3}{2}}\Big)\ x^{\frac{5}{2}} - a^{\frac{1}{2}}x^2 - 4ax^{\frac{3}{2}} + 6a^{\frac{3}{2}}x - 2a^2x^{\frac{1}{2}}\ \Big(x - a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\begin{array}{l} x^{\frac{5}{2}} \qquad\qquad - 4ax^{\frac{3}{2}} + 2a^{\frac{3}{2}}x \\ \hline \quad - a^{\frac{1}{2}}x^2 \qquad + 4a^{\frac{3}{2}}x - 2a^2x^{\frac{1}{2}} \\ \quad - a^{\frac{1}{2}}x^2 \qquad + 4a^{\frac{3}{2}}x - 2a^2x^{\frac{1}{2}} \end{array}$$

It is well to observe that no algebraic operation with homogeneous quantities can destroy the homogeneity (59), which will be found existing throughout in all the products, remainders, quotients, &c. Moreover, in all such products and quotients, the Law of Dimensions will be observed, as indicated by the formulæ, $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$: thus, in Ex. 2, the quotient is of $\frac{5}{2} - \frac{3}{2} = 1$ dimension, and all the products of $\frac{3}{2} + 1 = \frac{5}{2}$ dimensions. This observation will often help us to detect errors in Mult^n, Div^n &c., especially in dealing with fractional indices.

Ex. 49.

1. Simplify $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$, $\sqrt[3]{a^2\sqrt{a^{-1}}}$, $\sqrt{a^{-1}\sqrt{a^3\sqrt{a^{-4}}}}$, $\{\sqrt{a^2b\sqrt[3]{a^{-4}b^{-2}}}\}^6$.
2. Simplify $\{x^{-\frac{3}{2}}y.(xy^{-2})^{-\frac{1}{2}}.(x^{-1}y)^{-\frac{2}{3}}\}^3$, $\{x^{\frac{1}{2}}y^{-\frac{1}{4}}\sqrt{(x^{\frac{1}{3}}y^{\frac{1}{2}}\sqrt{y^{\frac{1}{3}}})}\}^3$.
3. Simplify $\sqrt[12]{\{\sqrt[3]{xy^2\sqrt[2]{xyz^4\sqrt[4]{x^{-1}y^{-2}z^{-3}}}}\}^{12}}$, $\sqrt[m]{a^{2m-n}b^{5m+1}c^{3p}} \times \sqrt[m]{a^n b^{m-1} c^{m-3p}}$.
4. Multiply $x + 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$ by $x - 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$.
5. Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.
6. Multiply $a^{\frac{5}{2}} - 2a^2b^{\frac{1}{3}} + 4a^{\frac{3}{2}}b^{\frac{2}{3}} - 8ab + 16a^{\frac{1}{2}}b^{\frac{4}{3}} - 32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}} + 2b^{\frac{1}{3}}$
7. Multiply $a^{\frac{3}{2}} - a^{\frac{2}{3}} + a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$ by $a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$.
8. Multiply $x^{\frac{5}{8}} + x^{\frac{1}{2}}y^{-\frac{1}{8}} + x^{\frac{3}{8}}y^{-\frac{1}{4}} + x^{\frac{1}{4}}y^{-\frac{3}{8}} + x^{\frac{1}{8}}y^{-\frac{1}{2}} + y^{-\frac{5}{8}}$
   by $x^{\frac{3}{8}} - x^{\frac{1}{4}}y^{-\frac{1}{8}} + x^{\frac{1}{8}}y^{-\frac{1}{4}} - y^{-\frac{3}{8}}$.

9. Divide $16x - y^2$ by $2x^{\frac{1}{4}} - y^{\frac{1}{2}}$, and $x^{-1} - y^{-1}$ by $x^{\frac{1}{3}} - y^{-\frac{1}{3}}$.

10. Divide $a^{-3} - 64b^2$ by $a^{-\frac{1}{2}} + 2b^{\frac{1}{3}}$, and $x - 2x^{\frac{1}{2}} + 1$ by $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} + 1$.

11. Divide $8a^{\frac{3}{2}} + b^{-\frac{3}{2}} - c + 6a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{3}}$ by $2a^{\frac{1}{2}} + b^{-\frac{1}{2}} - c^{\frac{1}{3}}$.

12. Find the cubes of $a^{\frac{1}{3}}b^{-1} + a^{-\frac{1}{3}}b$ and $\frac{2}{3}x^{\frac{2}{3}}y^{-\frac{1}{3}} - \frac{3}{2}x^{-\frac{1}{3}}y^{\frac{2}{3}}$.

13. Find the cube of $a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{6}} + 3b^{\frac{1}{3}}$.

14. Write down the square of $a^{\frac{2}{3}} - 2a^{\frac{1}{3}} + 3 - 2a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$.

15. Find the fourth and fifth powers of $x^{\frac{3}{4}} - y^{\frac{5}{2}}$, and $a^{\frac{1}{2}}b^{-\frac{1}{2}} - a^{-\frac{1}{2}}b^{\frac{1}{2}}$.

16. Find the square root of $a^2b^2 + 2ab^{-1} + 3 + 2a^{-1}b + a^{-2}b^2$.

17. Find the square root of

$$a^{\frac{4}{3}} - 3a + \frac{33}{4}a^{\frac{2}{3}} - 21a^{\frac{1}{3}} + 45 - 63a^{-\frac{1}{3}} + 90a^{-\frac{2}{3}} - 108a^{-1} + 81a^{-\frac{4}{3}}.$$

18. Find the cube root of

$$a^{-\frac{3}{2}}x^{\frac{3}{2}} - 3a^{-1}x + 6a^{-\frac{1}{2}}x^{\frac{1}{2}} - 7 + 6a^{\frac{1}{2}}x^{-\frac{1}{2}} - 3x^{-1} + a^{\frac{3}{2}}x^{-\frac{3}{2}}.$$

19. Find the fourth root of $x^4y^{-\frac{4}{3}} - 4x^{\frac{5}{2}}y^{-\frac{1}{3}} + 6xy^{\frac{2}{3}} - 4x^{-\frac{1}{2}}y^{\frac{5}{3}} + x^2y^{\frac{8}{3}}$.

20. Find the fourth root of $16x^6 - 96x^{\frac{9}{2}}y^{\frac{3}{4}} + 216x^3y^{\frac{3}{2}} - 216x^{\frac{3}{2}}y^{\frac{9}{4}} + 81y^3$.

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108. Since every fractional index indicates by its denominator a root to be extracted, all quantities having such indices are expressed as surds.

When a *negative* quantity has the denominator of its index (reduced to its lowest terms) *even* (46), the expression will be imaginary.

thus $\sqrt{-3}$ or $(-3)^{\frac{1}{2}}$, $\sqrt[4]{-9}$ or $(-9)^{\frac{1}{4}}$, are imaginary quantities; but $(-4)^{\frac{4}{6}}$ is not so, since it is the same as $(-4)^{\frac{2}{3}}$, where the root to be taken is *odd*.

109. In the case of a *numerical* surd, expressed with a fractional index, should the numerator be any other than *unity*, we may take at once the required power and so have unity only for the numerator, and a simple root to be extracted.

Thus $2^{\frac{2}{3}} = (2^2)^{\frac{1}{3}} = 4^{\frac{1}{3}}$ or $\sqrt[3]{4}$, $3^{-\frac{3}{5}} = (3^{-3})^{\frac{1}{5}} = (\frac{1}{27})^{\frac{1}{5}}$ or $\sqrt[6]{\frac{1}{27}}$.

110. Quantities are often expressed in the *form* of surds, which are not really so, *i. e.* when we *can*, if we please, extract the roots indicated.

Thus $\sqrt{a}$, $\sqrt[3]{7}$, $(a^2+ab+b^2)^{\frac{1}{3}}$ are *actually* surds, whose roots we cannot obtain; but $\sqrt{a^2}$, $\sqrt[3]{27}$, $(4a^2+4ab+b^2)^{\frac{1}{2}}$ are only *apparently* so, and are respectively equivalent to a, 3, $2a+b$.

Conversely, any rational quantity may be expressed in the form of a surd, by raising it to the power indicated by the denominator of the surd-index.

Thus $2=4^{\frac{1}{2}}=\sqrt[3]{8}=\&c.$, $a=\sqrt[3]{a^3}$, $\frac{3}{2}c=(\frac{9}{4}c^2)^{\frac{1}{2}}$, $a+x=(a^2+2ax+x^2)^{\frac{1}{2}}$

111. In like manner a *mixed* surd, *i. e.* a product partly rational and partly surd, may be expressed as an entire surd, by raising the rational factor to the power indicated by the denominator of the surd-index, and placing beneath the sign of Evolution the product of this power and the surd-factor.

Thus $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{12}$, $3.2^{\frac{2}{3}}=3\sqrt[3]{4}=\sqrt[3]{27}\times\sqrt[3]{4}=\sqrt[3]{108}$.

$$2a\sqrt{b}=\sqrt{4a^2b},\ 4a\sqrt[3]{\frac{c}{2a}}=\sqrt[3]{\frac{64a^3c}{2a}}=\sqrt[3]{32a^2c}.$$

Conversely, a surd may often be reduced to a mixed form, by separating the quantity beneath the sign of Evolution into factors, of one of which the root required may be obtained, and set outside the sign.

$$\text{Thus } \sqrt{20}=\sqrt{4\times5}=2\sqrt{5},\ \sqrt[3]{24}=\sqrt[3]{8\times3}=2\sqrt[3]{3},$$
$$\sqrt{\tfrac{48}{25}a^3b}=\tfrac{4}{5}a\sqrt{3ab},\ \sqrt[4]{\tfrac{32}{81}a^5b^4c^3}=\tfrac{2}{3}ab\sqrt[4]{2ac^3}.$$

112. A surd is reduced to its simplest form, when the quantity beneath the root, or *surd-factor*, is made as *small* as possible, but so as still to remain *integral*.

Hence, if the surd-factor be a *fraction*, its num^r and den^r should both be multiplied by such a number, as will allow us to take the latter from under the root.

Thus $\sqrt{\frac{2}{3}}=\sqrt{\frac{2.3}{3^2}}=\frac{1}{3}\sqrt{6}$, $\frac{5}{2}\sqrt[3]{\frac{24}{5}}=5\sqrt[3]{\frac{3}{5}}=5\sqrt[3]{\frac{3.5^2}{5^3}}=\sqrt[3]{75}$.

These latter forms allow of our calculating more easily the numerical values of the surd quantities. Thus to find that of $\sqrt{\frac{2}{3}}$, we should have had to extract both $\sqrt{2}$ and $\sqrt{3}$, and then to divide the one by the other, a tedious process, since each would be expressed by decimals that do not terminate; whereas in $\frac{1}{3}\sqrt{6}$, we have only to find $\sqrt{6}$, and divide this by the integer, 3.

Similar surds are those which have, or may be made to have, the *same* surd-factors.

Thus, $3\sqrt{a}$ and $\sqrt{a}$, $2a\sqrt[3]{c}$ and $3b^3\sqrt[3]{c}$, are pairs of similar surds; and $\sqrt{8}$, $\sqrt{50}$, $\sqrt{18}$ are also similar, because they may be written $2\sqrt{2}$, $5\sqrt{2}$, $3\sqrt{2}$.

Ex. 50.

1. Express $4^{\frac{3}{4}}$, $9^{\frac{2}{5}}$, $3^{-\frac{2}{3}}$, $2^{-\frac{3}{4}}$, $(\frac{2}{3})^{-\frac{1}{4}}$, $(\frac{1}{2})^{-\frac{3}{5}}$ with indices, whose numerator is unity.

2. Express 5, $2\frac{1}{2}$, $\frac{2}{3}a$, $\frac{3}{2}a^2$, $\frac{1}{2}(a+b)$, as surds, with indices $\frac{1}{2}$ and $\frac{1}{3}$.

3. Express 3^{-2}, $(3\frac{1}{3})^{-1}$, a^{-2}, $ab^{-1}c^{-2}$, with indices $\frac{1}{3}$ and $-\frac{1}{4}$.

Reduce to entire surds

4. $5\sqrt{5}$, $2\sqrt{\frac{3}{4}}$, $\frac{2}{3}.3^{\frac{3}{2}}$, $\frac{3}{5}\sqrt{1\frac{2}{3}}$, $\frac{1}{2}(\frac{3}{4})^{-\frac{1}{2}}$, $25\,(1\frac{1}{4})^{-\frac{3}{2}}$.

5. $3\sqrt[3]{2}$, $8.2^{-\frac{1}{3}}$, $4.2^{\frac{3}{4}}$, $3.3^{-\frac{3}{4}}$, $\frac{2}{3}(\frac{2}{3})^{-\frac{2}{3}}$, $\frac{1}{2}(\frac{3}{4})^{-\frac{2}{5}}$.

6. $2\sqrt{a}$, $7a\sqrt{2x}$, $a\,(ab)^{-1}$, $(a+b)\,(a^2-b^2)^{-\frac{1}{2}}$, $(a-b)\,(a^3-b^3)^{-1}$.

7. $a\sqrt{\dfrac{2b}{a}}$, $3ax\sqrt{\dfrac{2a}{3x}}$, $\dfrac{2a}{3b}\sqrt[3]{\dfrac{3b}{2a}}$, $\dfrac{2a}{3}\sqrt[3]{\dfrac{9}{4a^2}}$, $(a+x)\sqrt{\dfrac{a-x}{a+x}}$.

Reduce to their simplest form

8. $\sqrt{45}$, $\sqrt{125}$, $3\sqrt{432}$, $\sqrt[3]{135}$, $3\sqrt[3]{432}$, $\sqrt{\frac{3}{2}}$, $2\sqrt[3]{\frac{3}{2}}$, $3\sqrt[4]{\frac{2}{3}}$, $4\sqrt[3]{3\frac{3}{8}}$.

9. $8^{\frac{3}{4}}$, $32^{\frac{2}{3}}$, $72^{\frac{3}{5}}$, $(1\frac{1}{8})^{-\frac{1}{2}}$, $(20\frac{1}{4})^{-\frac{3}{4}}$, $(30\frac{3}{8})^{-\frac{2}{5}}$, $\frac{3}{2}\sqrt{\frac{1}{7}}$, $5\sqrt[3]{4\frac{1}{20}}$, $\frac{5}{2}\sqrt[4]{9\frac{3}{5}}$.

10. Shew that $\sqrt{12}$, $3\sqrt{75}$, $\frac{1}{2}\sqrt{147}$, $\frac{2}{3}\sqrt{\frac{4}{75}}$, $\sqrt[4]{\frac{9}{16}}$, and $(144)^{-\frac{1}{4}}$ are similar surds.

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113. To compare surds with one another in magnitude, express them as *entire* surds, and then reduce their indices, if necessary, to a common denominator, simplifying as in (109): their relative values will be now apparent.

Thus $3\sqrt{2}$ and $2\sqrt{3}$, expressed as entire surds, are $\sqrt{18}$ and $\sqrt{12}$, and it is at once plain which is greatest: but $3\sqrt{2}$ and $2\sqrt[3]{3}$, or their equivalents $\sqrt{18}$ and $\sqrt[3]{24}$, in which *different* roots are to be taken, cannot be at once compared; here then $18^{\frac{1}{2}}=18^{\frac{3}{6}}=\sqrt[6]{5832}$, and $24^{\frac{1}{3}}=24^{\frac{2}{6}}=\sqrt[6]{576}$, and now their comparative values are evident.

114. To *add* or *subtract* surds, reduce them, when similar, to the same surd-factor, and add or subtract their rational factors.

Thus $\sqrt{8}+\sqrt{50}-\sqrt{18}=2\sqrt{2}+5\sqrt{2}-3\sqrt{2}=4\sqrt{2}$,
$4a\sqrt[3]{a^3b^4}+b\sqrt[3]{8a^6b}-\sqrt[3]{125a^6b^4}=4a^2b\sqrt[3]{b}+2a^2b\sqrt[3]{b}-5a^2b\sqrt[3]{b}=a^2b\sqrt[3]{b}$.

Dissimilar surds can only be connected by their signs.

115. To *multiply* surds, reduce them (113) to the same surd-index, and multiply separately the rational and surd-factors, retaining the same surd-index for the product of the latter.

Thus $\sqrt{8}\times3\sqrt{2}=3\sqrt{16}=12$, $2\sqrt{3}\times3\sqrt{10}\times4\sqrt{6}=24\sqrt{180}=144\sqrt{5}$
$2\sqrt{3}\times3\sqrt[3]{2}=2\sqrt[6]{27}\times3\sqrt[6]{4}=6\sqrt[6]{108}$.

Compound surd quantities are multiplied according to the method of rational quantities.

Ex. 1. $(2\pm\sqrt{3})^2=4\pm4\sqrt{3}+3=7\pm4\sqrt{3}$.
Ex. 2. $(2+\sqrt{3})(2-\sqrt{3})=4-3=1$.
Ex. 3. $(2+\sqrt{3})(3-\sqrt{2})=6+3\sqrt{3}-2\sqrt{2}-\sqrt{6}$.
Ex. 4. $(1+\sqrt{2})^4=1+4\sqrt{2}+12+8\sqrt{2}+4=17+12\sqrt{2}$.

116. *Division of surds* is performed, when the divisor is a simple quantity, by a process similar to that for multiplication.

Thus $(8\sqrt{2}-12\sqrt{3}+3\sqrt{6}-4)\div2\sqrt{6}=4\sqrt{\tfrac{2}{6}}-6\sqrt{\tfrac{3}{6}}+\tfrac{3}{2}-\dfrac{2}{\sqrt{6}}$
$=\tfrac{4}{3}\sqrt{3}-3\sqrt{2}+\tfrac{3}{2}-\tfrac{1}{3}\sqrt{6}$,
$(2\sqrt{3}-6\sqrt[3]{2})\div\sqrt{6}=2\sqrt{\tfrac{3}{6}}-6\sqrt[6]{\tfrac{4}{216}}=\sqrt{2}-\sqrt[6]{864}$.

117. But, if the divisor be *compound*, the division is not so easily performed. The form, however, in which compound surds usually occur, is that of a *binomial quadratic* surd, *i. e.*, a binomial, one or both of whose terms are surds, in which the *square* root is to be taken, such as $3 + 2\sqrt{5}$, $2\sqrt{3} - 3\sqrt{5}$, or, generally, $\sqrt{a} \pm \sqrt{b}$, where one or both terms may be irrational; and it will be easy, in such a case, to convert the operation of division into one of multiplication, by putting the dividend and divisor in the form of a fraction, and multiplying both num^r and den^r by that quantity, which is obtained by changing the signs between the two terms of the den^r. By this means the den^r will be made *rational*: thus, if it be originally of the form $\sqrt{a} \pm \sqrt{b}$, it will become a rational quantity, $a-b$, when both num^r and den^r are multiplied by $\sqrt{a} \pm \sqrt{b}$.

Ex. 1. $\dfrac{2+\sqrt{3}}{3+\sqrt{3}} = \dfrac{(2+\sqrt{3})\,(3-\sqrt{3})}{(3+\sqrt{3})\,(3-\sqrt{3})} = \dfrac{6+3\sqrt{3}-2\sqrt{3}-3}{9-3} = \dfrac{3+\sqrt{3}}{6}$.

Ex. 2. $\dfrac{1}{2\sqrt{2}-\sqrt{3}} = \dfrac{2\sqrt{2}+\sqrt{3}}{8-3} = \dfrac{2\sqrt{2}+\sqrt{3}}{5}$.

Fractions thus modified are considered to be reduced to their simplest form, for the reason mentioned in (112).

Ex. **51.**

1. Compare $6\sqrt{3}$ and $4\sqrt{7}$; $3\sqrt[3]{3}$ and $2\sqrt[3]{10}$; $2\sqrt[3]{15}$, $4\sqrt[3]{2}$, and $3\sqrt[3]{5}$; $\sqrt{5}$ and $\sqrt[3]{11}$; $\frac{1}{2}\sqrt{2}$ and $\frac{1}{3}\sqrt[4]{27}$; $\sqrt{5}$, $2\sqrt[3]{\frac{3}{2}}$, and $3\,(4\frac{1}{2})^{-\frac{1}{6}}$

2. Simplify $\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18}$, $\sqrt[3]{40} - \frac{1}{2}\sqrt[3]{320} + \sqrt[3]{135}$.

3. Simplify $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}}$, $\sqrt[3]{72} - 3\sqrt[3]{\frac{1}{3}} + 6\sqrt[3]{21\frac{1}{3}}$.

4. Multiply $3\sqrt{8}$ by $2\sqrt{6}$, $3\sqrt{15}$ by $4\sqrt{20}$, and $2\sqrt[3]{4}$ by $3\sqrt[3]{54}$.

5. Find the continued product of $3\sqrt{8}$, $2\sqrt[3]{6}$, and $3\sqrt[4]{54}$; and of $2\sqrt{24}$, $3\sqrt[4]{18}$, and $4\sqrt[6]{24}$.

6. Multiply $3\sqrt{3}+2\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$, and $2\sqrt{15}-\sqrt{6}$ by $\sqrt{5}+2\sqrt{2}$.

7. Find the continued product of $4+2\sqrt{2}$, $1-\sqrt{3}$, $4-2\sqrt{2}$, $\sqrt{2}+\sqrt{3}$, $1+\sqrt{3}$, and $\sqrt{2}-\sqrt{3}$.

8. Div. $2\sqrt{3}+3\sqrt{2}+\sqrt{30}$ by $3\sqrt{6}$, and $2\sqrt{3}+3\sqrt[3]{2}+\sqrt[4]{30}$ by $3\sqrt{2}$.

9. Rationalize the denominators of

$$\frac{1}{2\sqrt{2}-\sqrt{3}},\quad \frac{4}{\sqrt{5}-1},\quad \frac{3}{\sqrt{5}+\sqrt{2}},\quad \frac{8-5\sqrt{2}}{3-2\sqrt{2}},\quad \frac{3+\sqrt{5}}{3-\sqrt{5}},\quad \frac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}.$$

10. Divide $2+4\sqrt{7}$ by $2\sqrt{7}-1$, $3+2\sqrt{5}$ by $2\sqrt{5}-1$, and $5-2\sqrt{6}$ by $6-2\sqrt{6}$.

Simplify

11. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}},\quad \dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}},\quad \dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}},\quad \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}.$

12. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}, \& \dfrac{1}{4(1+\sqrt{x})}+\dfrac{1}{4(1-\sqrt{x})}+\dfrac{1}{2(1+x)}.$

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118. The following facts should be noticed.

(i) *The product of two dissimilar surds cannot be rational.*

Let $\sqrt{x}\times\sqrt{y}=m$, a rational quantity; $\therefore xy=m^2$;

$$\text{hence } y=\frac{m^2}{x}=\frac{m^2}{x^2}x, \text{ and } \sqrt{y}=\frac{m}{x}\sqrt{x},$$

or $\sqrt{y}$ may be made to have the same surd-factor as $\sqrt{x}$; that is, $\sqrt{x}$ and $\sqrt{y}$ must be *similar* surds (112).

(ii) *A surd cannot equal the sum or difference of a rational quantity and a surd, or of two dissimilar surds.*

For let $\sqrt{a}=x\pm\sqrt{y}$, $\therefore a=x^2\pm 2x\sqrt{y}+y$;

whence $\pm 2x\sqrt{y}=a-x^2-y$, and $\pm\sqrt{y}=\dfrac{a-x^2-y}{2x}$,

or a surd = a rational quantity, which is absurd.

Again, let $\sqrt{a}=\sqrt{x}\pm y$, $\therefore a=x\pm 2\sqrt{xy}+y$, whence $\pm 2\sqrt{xy}=a-x-y$, and $\pm\sqrt{xy}=\frac{1}{2}(a-x-y)$, or the product of two dissimilar surds = a rational quantity, which is impossible.

(iii) *If* $a+\sqrt{b}=x+\sqrt{y}$, *then* $a=x$, *and* $\sqrt{b}=\sqrt{y}$.

For since $a+\sqrt{b}=x+\sqrt{y}$, we have $\sqrt{b}=(x-a)+\sqrt{y}$; so that, if x be not equal to a, we shall have $\sqrt{b}=$sum of a rational quantity and a surd, which is impossible; hence $x = a$, and $\therefore \sqrt{b} = \sqrt{y}$.

Hence also, if $a + \sqrt{b} = x + \sqrt{y}$, then $a - \sqrt{b} = x - \sqrt{y}$; and, if $a + \sqrt{b} = 0$, we must have separately $a = 0$, and $b = 0$; otherwise we should have $\sqrt{b} = -a$, or a surd = a rational quantity.

(iv) *If* $\sqrt{a + \sqrt{b}} = x + \sqrt{y}$, *then* $\sqrt{a - \sqrt{b}} = x - \sqrt{y}$.

For since $\sqrt{a + \sqrt{b}} = x + \sqrt{y}$, we have, squaring,

$a + \sqrt{b} = x^2 + 2x\sqrt{y} + y$; $\therefore a = x^2 + y$, and $\sqrt{b} = 2x\sqrt{y}$;

whence $a - \sqrt{b} = x^2 - 2x\sqrt{y} + \sqrt{y}$, and $\sqrt{a - \sqrt{b}} = x - \sqrt{y}$.

So also, if $\sqrt{a} + \sqrt{b} = \sqrt{x} + \sqrt{y}$, then $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

119. *To extract the square root of a binomial surd, one of whose terms is rational, the other a quadratic surd.*

Let $a + \sqrt{b}$ represent the given surd;

assume $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, $\therefore \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$;

hence, multiplying these equations, $\sqrt{a^2 - b} = x - y$;

but, since $a + \sqrt{b} = x + y + 2\sqrt{xy}$, $\therefore$ also (118, iii) $a = x + y$:

$\therefore$, adding and subtracting, $a + \sqrt{a^2 - b} = 2x$, $a - \sqrt{a^2 - b} = 2y$:

$$\therefore x = \tfrac{1}{2}(a + \sqrt{a^2 - b}),\ y = \tfrac{1}{2}(a - \sqrt{a^2 - b}),$$

$$\&\ \sqrt{(a \pm \sqrt{b})} = \sqrt{x} \pm \sqrt{y} = \sqrt{\{\tfrac{1}{2}(a + \sqrt{a^2 - b})\}} \pm \sqrt{\{\tfrac{1}{2}(a - \sqrt{a^2 - b})\}}.$$

Ex. Find the square root of $7 \pm 2\sqrt{10}$.

Let $\sqrt{7 + 2\sqrt{10}} = \sqrt{x} + \sqrt{y}$, $\therefore \sqrt{7 - 2\sqrt{10}} = \sqrt{x} - \sqrt{y}$;

and $\sqrt{49 - 40} = x - y$, whence $3 = x - y$;

but, since $7 + 2\sqrt{10} = x + y + 2\sqrt{xy}$, $\therefore$ also $7 = x + y$;

$\therefore 10 = 2x$, $4 = 2y$, or $x = 5$, $y = 2$; and $\sqrt{7 \pm 2\sqrt{10}} = \sqrt{5} \pm \sqrt{2}$.

Ex. 52.

Find the square roots of

1. $4 + 2\sqrt{3}$.
2. $11 + 6\sqrt{2}$.
3. $8 - 2\sqrt{15}$.
4. $38 - 12\sqrt{10}$.
5. $41 - 24\sqrt{2}$.
6. $2\frac{1}{4} - \sqrt{5}$.
7. $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$.
8. $\frac{137}{144} - \frac{1}{3}\sqrt{2}$.

Find the fourth roots of

9. $17 + 12\sqrt{2}$.
10. $56 - 24\sqrt{5}$.
11. $\frac{3}{2}\sqrt{5} + 3\frac{1}{2}$.
12. $48\frac{1}{16} + \frac{23}{2}\sqrt{15}$.

CHAPTER IX.

QUADRATIC EQUATIONS.

120. Some equations involving surds are reducible to simple equations, as in the following examples.

Ex. 1. $\sqrt{12+x} = 2 + \sqrt{x}$.

Squaring, we have $12+x=4+4\sqrt{x}+x$ $\therefore 4\sqrt{x}=8$, and $\sqrt{x}=2$, or $x=4$.

Ex. 2. $3 + x - \sqrt{x^2+9} = 2$.

Here $\sqrt{x^2+9} = 1 + x$: [observe in other similar cases to take this step, when possible, by which we get the surd *by itself* on one side, and so it will disappear upon squaring :]

hence $x^2 + 9 = 1 + 2x + x^2$, and $x = 4$.

Ex. 53.

1. $\sqrt{5(x+2)} = \sqrt{5x} + 2$.
2. $\sqrt{x} - \sqrt{a+x} = \sqrt{\dfrac{a}{x}}$.
3. $\sqrt{xb} + \sqrt{b(a+x)} = x^{\frac{1}{2}}$.
4. $\sqrt{bx+x^2} = 1 + x$.
5. $\frac{1}{3}\sqrt[3]{17x-26} + \frac{3}{8} = 1\frac{1}{24}$.
6. $a + x - \sqrt{a^2+x^2} = b$.
7. $\sqrt{x-a} = \sqrt{x+\sqrt{b+x}}$.
8. $-\sqrt{x} + \sqrt{x+2\sqrt{ax+a^2}} = \sqrt{a}$.
9. $a + x - \sqrt{2ax+x^2} = b$.
10. $a + x + \sqrt{a^2+bx+x^2} = b$.

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121. *Quadratic Equations* are those in which the *square* of the unknown quantity is found. Of these there are two species :

(i) *Pure* Quadratics, in which the square only is found, without the first power, as $x^2 - 9 = 0$, &c. ;

(ii) *Adfected* Quadratics, where the first power enters as well as the square, as $x^2 - 3x + 2 = 0$, &c.

122. Pure Quadratics are solved, as in simple equations, by collecting the unknown quantities on one side, and the known quantities on the other. We shall thus find the value of x^2, and thence the value of x, to which we must prefix the double sign ($\pm$).

Such equations therefore will have two equal roots, with contrary signs.

Ex. 1. $x^2 - 9 = 0$. Here $x^2 = 9$, and $x = \pm 3$.

If we had put $\pm x = \pm 3$, we should still have had only these *two* different values of x, viz. $x = +3$, $x = -3$; since $-x = +3$ gives $x = -3$, and $-x = -3$ gives $x = +3$.

Ex. 2. $\frac{1}{8}(3x^2 + 5) - \frac{1}{3}(x^2 + 21) = 39 - 5x^2$.

Reducing, $121x^2 = 1089$; $\therefore x^2 = 9$, and $x = \pm 3$.

Ex. 3. $\dfrac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2} - x} = \dfrac{b}{c}$. Here (85. vi) $\dfrac{\sqrt{a^2 + x^2}}{x} = \dfrac{b + c}{b - c}$;

$\therefore \dfrac{a^2 + x^2}{x^2} = \left(\dfrac{b + c}{b - c}\right)^2$, and $\dfrac{x^2}{a^2} = \dfrac{(b - c)^2}{4bc}$, or $x = \pm \dfrac{a(b - c)}{2\sqrt{bc}}$.

The above method of reduction from (85. vi) may always be applied with advantage to an equation of the above form, when the unknown quantity does not enter in *both* sides of it.

Ex. 54.

1. $\frac{1}{2}x^2 = 14 - 3x^2$.
2. $x^2 + 5 = \frac{10}{3}x^2 - 16$.
3. $(x+2)^2 = 4x + 5$.
4. $\dfrac{3}{1 + x} + \dfrac{3}{1 - x} = 8$.
5. $\dfrac{3}{4x^2} - \dfrac{1}{6x^2} = \dfrac{7}{3}$.
6. $8x + \dfrac{7}{x} = \dfrac{65x}{7}$.
7. $\dfrac{3x^2}{4} - \dfrac{15x^2 + 8}{6} = 2x^2 - 3$.
8. $\dfrac{x^2}{5} - \dfrac{x^2 - 10}{15} = 7 - \dfrac{50 + x^2}{25}$.
9. $\dfrac{3x^2 - 27}{x^2 + 3} + \dfrac{90 + 4x^2}{x^2 + 9} = 7$.
10. $\dfrac{4x^2 + 5}{10} - \dfrac{2x^2 - 5}{15} = \dfrac{7x^2 - 25}{20}$.
11. $\dfrac{10x^2 + 17}{18} - \dfrac{12x^2 + 2}{11x^2 - 8} = \dfrac{5x^2 - 4}{9}$.
12. $\dfrac{14x^2 + 16}{21} - \dfrac{2x^2 + 8}{8x^2 - 11} = \dfrac{2x^2}{3}$.
13. $\dfrac{2}{x + \sqrt{2 - x^2}} + \dfrac{2}{x - \sqrt{2 - x^2}} = x$.
14. $\dfrac{1}{a - \sqrt{a^2 - x^2}} - \dfrac{1}{a + \sqrt{a^2 - x^2}} = \dfrac{a}{x^2}$.
15. $\dfrac{\sqrt{a^2 - x^2} - \sqrt{b^2 + x^2}}{\sqrt{a^2 - x^2} + \sqrt{b^2 + x^2}} = \dfrac{c}{d}$.
16. $x + \sqrt{a^2 + x^2} = \dfrac{na^2}{\sqrt{a^2 + x^2}}$.

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123. An *adfected* quadratic may always be reduced to the form, $x^2 + px + q = 0$, where the coeff. of x^2 is $+1$, and p, q, represent numbers or known quantities.

Now, in this equation, we have $x^2 + px = -q$, and adding $(\frac{1}{2}p)^2$ to each side, we get $x^2 + px + \frac{1}{4}p^2 = \frac{1}{4}p^2 - q$:

by this step, the first side becomes a complete square; and taking the square root of each side, prefixing, as before, the double sign to that of the latter, we have

$$x+\tfrac{1}{2}p=\pm\sqrt{\tfrac{1}{4}p^2-q}, \text{ and } x=-\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-q};$$

which expression gives us, according as we take the upper or lower sign, two roots of the quadratic.

124. From the preceding we derive the following Rule for the solution of an adfected quadratic:

Reduce it to its simplest form; set the terms involving x^2 and x on one side, (the coeff. of x^2 being $+1$,) and the known quantity on the other; then, if we *add the square of half the coeff. of* x *to each side*, the first will become a complete square; and taking the square root of each, prefixing the double sign to the second, we shall obtain, as above, the two roots of the equation.

Ex. 1. $x^2-6x=7$. Here $x^2-6x+9=7+9=16$;
whence $x-3=\pm 4$, and $x=3+4=7$, or $x=3-4=-1$;
so that 7 and -1 are the two roots of the equation.

Ex. 2. $x^2+14x=95$. Here $x^2+14x+49=95+49=144$;
whence $x+7=\pm 12$, and $x=-7+12=5$, or $x=-7-12=-19$.

Ex. **55**.

1. $x^2-2x=8$.	2. $x^2+10x=-9$.	3. $x^2-14x=120$.
4. $x^2-12x=-35$.	5. $x^2+32x=320$.	6. $x^2+100x=1100$.

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125. If the coefficient of x be *odd*, its half will be a fraction. In adding its square to the *first* side, we may *express* the squaring, without effecting it, by means of a bracket.

Ex. 1. $x^2-5x=-6$. Here $x^2-5x+(\frac{5}{2})^2=-6+\frac{25}{4}=\frac{1}{4}(-24+25)=\frac{1}{4}$;
whence $x-\frac{5}{2}=\pm\frac{1}{2}$, and $x=\frac{5}{2}+\frac{1}{2}=\frac{6}{2}=3$, or $x=\frac{5}{2}-\frac{1}{2}=\frac{4}{2}=2$.

Ex. 2. $x^2-x=\frac{3}{4}$. Here $x^2-x+(\frac{1}{2})^2=\frac{3}{4}+\frac{1}{4}=1$;
whence $x-\frac{1}{2}=\pm 1$ and $x=\frac{1}{2}+1=1\frac{1}{2}$, or $x=\frac{1}{2}-1=-\frac{1}{2}$.

Ex. **56.**

1. $x^2 + 7x = 8$. 2. $x^2 - 13x = 68$. 3. $x^2 + 25x = -100$.
4. $x^2 + 13x = -12$. 5. $x^2 + 19x = 20$ 6. $x^2 + 111x = 3400$.

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126. If the coefficient of x be a fraction, its half will, of course, be found by halving the numerator, if possible—if not, by doubling the denominator.

Ex. 1. $x^2 + \frac{10}{3}x = 19$. Here $x^2 + \frac{10}{3}x + (\frac{5}{3})^2 = 19 + \frac{25}{9} = \frac{196}{9}$; whence $x + \frac{5}{3} = \pm\frac{14}{3}$, and $x = -\frac{5}{3} + \frac{14}{3} = 3$, or $x = -\frac{5}{3} - \frac{14}{3} = -6\frac{1}{3}$.

Ex. 2. $x^2 + \frac{13}{5}x = 74$. Here $x^2 + \frac{13}{5}x + (\frac{13}{10})^2 = 74 + \frac{169}{100} = \frac{7569}{100}$; whence $x + \frac{13}{10} = \pm\frac{87}{10}$, and $x = -\frac{13}{10} + \frac{87}{10} = 7\frac{2}{5}$, or $x = -\frac{13}{10} - \frac{87}{10} = -10$.

Ex. **57.**

1. $x^2 - \frac{1}{3}x = 34$. 2. $x^2 - \frac{3}{2}x = 27$. 3. $x^2 + \frac{7}{5}x = 86$.
4. $x^2 - \frac{26}{7}x = 144$. 5. $x^2 + \frac{1}{12}x = 145$. 6. $x^2 - \frac{22}{13}x = 147$.

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127. In the following Examples the equations will first require reduction; and since the Rule requires that the coeff. of x^2 shall be $+1$, if it have any other coeff., we must first divide each term of the equation by it.

Ex. $3x^2 - 20x = 5$. Here $x^2 - \frac{20}{3}x = \frac{5}{3}$, and $x^2 - \frac{20}{3}x + \frac{100}{9} = \frac{115}{9}$; whence $x = \frac{1}{3}(10 \pm \sqrt{115})$, the roots being here *surd* quantities.

Ex. **58.**

1. $x = \frac{5}{3} + \frac{1}{12}x^2$. 2. $2x = 4 + \dfrac{6}{x}$. 3. $\frac{7}{11}x^2 - \frac{2}{3}x = \frac{1}{33}(11x + 18)$.

4. $11x^2 - 9x = 11\frac{1}{4}$. 5. $\frac{3}{4}(x^2 - 3) = \frac{1}{8}(x - 3)$. 6. $2x^2 + 1 = 11(x + 2)$.

. $x - \dfrac{x^3 - 8}{x^2 + 5} = 2$. 8. $\dfrac{1}{3} + \dfrac{1}{3 + x} + \dfrac{1}{3 + 2x} = 0$.

9. $\dfrac{x + 22}{3} - \dfrac{4}{x} = \dfrac{9x - 6}{2}$. 10. $\dfrac{x + 2}{x - 1} - \dfrac{4 - x}{2x} = 2\frac{1}{3}$.

11. $\dfrac{12}{5 - x} + \dfrac{4}{4 - x} = \dfrac{32}{x + 2}$. 12. $\dfrac{x}{x + 1} + \dfrac{x + 1}{x} = \dfrac{13}{6}$.

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128. An equation of the form $ax^2 + bx + c = 0$, or $ax^2 + bx = -c$ (where a, b, c, are any quantities whatever), may, however, be solved as follows, without dividing by the coefficient of x^2.

Multiply every term by $4a$, and add b^2 to each side,

then $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$, whence $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Ex. 1. $2x^2 - 7x + 3 = 0$, or $2x^2 - 7x = -3$. Here, mult. by $4 \times 2 = 8$, and add $7^2 = 49$ to each side; then $16x^2 - 56x + 49 = 49 - 24 = 25$; $\therefore 4x - 7 = \pm 5$, and $x = \frac{1}{4}(7 \pm 5) = 3$ or $\frac{1}{2}$.

The advanced student will find it well to accustom himself to apply at once (by memory) the formula above obtained for x.

Ex. 2. $(3x - 2)(1 - x) = 4$, or $3x^2 - 5x + 6 = 0$.

Here $x = \frac{1}{6}(5 \pm \sqrt{25 - 72}) = \frac{1}{6}\{5 \pm \sqrt{-47}\}$, the roots being *impossible.*

Ex. 59.

1. $\dfrac{1}{x-1} - \dfrac{1}{x+3} = \dfrac{1}{35}$.

2. $\dfrac{48}{x+3} = \dfrac{165}{x+10} - 5$.

3. $\dfrac{x+4}{3} - \dfrac{7-x}{x-3} = \dfrac{4x+7}{9} - 1$.

4. $\dfrac{3x-7}{x} + \dfrac{4x-10}{x+5} = 3\frac{1}{2}$.

5. $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = 8\frac{1}{3}$.

6. $\dfrac{2x+9}{9} + \dfrac{4x-3}{4x+3} = 3 + \dfrac{3x-16}{18}$

7. $\dfrac{5x}{x+4} - \dfrac{3x-2}{2x-3} = 2$.

8. $\dfrac{4x+7}{19} + \dfrac{5-x}{3+x} = \dfrac{4x}{9}$.

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129. The roots of $x^2 + px + q = 0$ are (123) $-\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$: hence, (i) if $\frac{1}{4}p^2 > q$, we shall have $\frac{1}{4}p^2 - q$ *positive*, and $\therefore \sqrt{\frac{1}{4}p^2 - q}$ a *possible* quantity: and since, in one root, it is taken with $+$, and in the other with $-$, the two roots will be *real and different* in value;

(ii) if $\frac{1}{4}p^2 = q$, we shall have $\frac{1}{4}p^2 - q = 0$, and, therefore, the two roots will be *real and equal* in value.

(iii) if $\frac{1}{4}p^2 < q$, we shall have $\frac{1}{4}p^2 - q$ *negative*, and $\sqrt{\frac{1}{4}p^2 - q}$ *impossible*, and so the two roots will be *impossible.*

Hence, if any equation be expressed in the form $x^2 + px + q = 0$, its roots will be *real and different*, *real and equal*, or *impossible*, according as $p^2 >, =,$ or $< 4q$.

So also in the more general equation, $ax^2 + bx + c = 0$, the roots will be *real and different*, *real and equal*, or *impossible*, according as $b^2 >, =,$ or $< 4ac$.

130. If a, β represent the two roots of $x^2+px+q=0$, then $-p=a+\beta$, and $q=a\beta$.

For $a=-\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-q}$, $\beta=-\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-q}$;

$\therefore a+\beta=-p$, and $a\beta=\frac{1}{4}p^2-(\frac{1}{4}p^2-q)=q$.

Hence, when any quadratic is reduced to the form $x^2+px+q=0$, we have

coeff. of 2nd term, with sign changed, $=$ *sum* of roots.

and 3rd term $=$ *product* of roots.

Thus, in (124), the equation, when expressed in this form, is $x^2-6x-7=0$, and the roots are there found, 7 and -1; and here $+6=7+(-1)=$ *sum* of roots, and $-7=7\times(-1)=$ *product* of roots.

So also $ax^2+bx+c=0$, expressed in this form, becomes

$x^2+\frac{b}{a}x+\frac{c}{a}=0$; $\therefore -\frac{b}{a}=$ *sum* of roots, $\frac{c}{a}=$ *product.*

131. *If a, β be the roots of $x^2+px+q=0$, then*

$$x^2+px+q=(x-a)(x-\beta).$$

For, (130) $x^2+px+q=x^2-(a+\beta)x+a\beta$

$$=x^2-ax-\beta x+a\beta=(x-a)(x-\beta).$$

So also if a, β be the roots of $ax^2+bx+c=0$, we have

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a(x-a)(x-\beta).$$

132. Hence we may form an equation with any given roots.

Thus with roots 2 and 3, we have $(x-2)(x-3)=x^2-5x+6=0$; with roots -2 and $\frac{1}{4}$, we have $(x+2)(x-\frac{1}{4})=x^2+\frac{7}{4}x-\frac{1}{2}=0$, or, clearing it of fractions, $4x^2+7x-2=0$.

This law is not confined to quadratics, but may be shewn to be true for equations of all dimensions.

Thus the biquadratic whose roots are $-1, 2, -2, 3$, is

$$(x+1)(x-2)(x+2)(x-3)=x^4-2x^3-7x^2+8x+12=0.$$

133. If one of the roots be 0, the corresponding factor will be $x-0$ or x.

Thus, with roots 0, 1, 3, we have $x(x-1)(x-3)=x^3-4x^2+3x=0$.

In such a case, then, x will occur in *every* term of the equation, and may therefore be struck out of each; but let it be noticed that, whenever we thus strike an x out of every term of an equation, it must not be neglected, since such an equation, as it originally stood, would be satisfied by $x=0$, which is therefore one of its roots.

Thus, in the above equation, we may strike an x out of every term, and thus reduce it to $x^2-4x+3=0$, which gives us the two roots, 1 and 3; but, besides these, we have the root $x=0$.

Ex. **60.**

Form the equation with roots

1. 7 and -3,
2. $\frac{2}{3}$ and $-\frac{3}{2}$
3. $3, -3, \frac{3}{4}, -\frac{3}{4}$.
4. 0, 1, 2, 3.
5. $0, -\frac{1}{2}, 1\frac{1}{2}, -1$.
6. $0, -1, 2, -2, \frac{1}{4}$.

We shall now give a few examples of quadratic equations of *two* unknowns. The solution of these is generally more difficult: but there are three cases of frequent occurrence, for which the following observations will be useful.

134. (i) Express, when possible, by means of one of the equations, either of the unknowns in terms of the other, and put this value for it in the other equation.

Ex. 1.
$$\left.\begin{aligned} x+\frac{1}{3}&=\frac{2x+y}{3} \quad &\text{(i)}\\ \frac{x+y}{x}&+\frac{4x-y}{2} \quad &\text{(ii)}\end{aligned}\right\}$$

From (i) we get $y=x+1$; and, putting this value for y in (ii), we have $\frac{2x+1}{x}=\frac{3x-1}{2}$, whence $x=2$ or $-\frac{1}{3}$, and $\therefore y=x+1=3$ or $\frac{2}{3}$.

The given equations have, therefore, two pairs of roots,

$x=2$ and $y=3$, or $x=-\frac{1}{3}$ and $y=\frac{2}{3}$.

135. (ii) When either of the two equations is *homogeneous* with respect to x and y, in all those terms of it which involve x and y, put $y=vx$, by which means we may generally without difficulty obtain an equation involving v only, which being determined, x and y may then be found.

Ex. 2. $$\left.\begin{array}{r} x^2+xy+y^2=7 \\ 2x+3y=8 \end{array}\right\} \quad \begin{array}{l}\text{(i)}\\ \text{(ii)}\end{array}$$

Here putting vx for y, $x^2(1+v+v^2)=7$, $\quad(\alpha)$

$$x(2+3v)=8; \quad (\beta)$$

$\therefore$ dividing (α) by the square of (β) the x^2 disappears, and we have

$$\frac{1+v+v^2}{(2+3v)^2}=\frac{7}{64}, \text{ whence } v=2 \text{ or } 18;$$

and from (β), $x(2+6)=8$, or $x=1$, and $y=vx=2$,

or $x(2+54)=8$, or $x=\frac{1}{7}$, and $y=vx=2\frac{4}{7}$,

(iii) When each of the two equations is *symmetrical* with respect to x and y, put $u+v$ for x and $u-v$ for y.

Def. An expression is said to be *symmetrical* with respect to x and y, when these quantities are similarly involved in it: thus $x^3+x^2y^2+y^3$, $\quad 4xy+5x+5y-1$, $\quad 2x^4-3x^2y-3xy^2+2y^4$, are symmetrical with respect to x and y.

Ex. 3. $$\left.\begin{array}{r} x^3+y^3=18xy \\ x+y=12 \end{array}\right\} \quad \begin{array}{l}\text{(i)}\\ \text{(ii)}\end{array}$$

Put $u+v$ for x, and $u-v$ for y;

then (i) becomes $(u+v)^3+(u-v)^3=18(u+v)(u-v)$,

or $u^3+3uv^2=9(u^2-v^2)$; $\quad(\alpha)$

and (ii) becomes $(u+v)+(u-v)=12$, whence $u=6$;

putting this for u in (α), $216+18v^2=9(36-v^2)$, whence $v=\pm 2$;

$\therefore x=u+v=6\pm 2=8$ or 4, and $y=u-v=6\pm 2=4$ or 8.

136. The preceding are *general* methods for the solution of equations of the kinds here referred to, and will sometimes succeed also in other equations; yet in many of these cases a little ingenuity will often suggest some step or artifice, by which the roots may be found more simply, but for which no rules can be given.

The methods pursued in the two following examples are worthy of notice in this respect.

Ex. 4. $\left.\begin{aligned} 3x^2 - 2xy &= 15 \\ 2x + 3y &= 12 \end{aligned}\right\}$ $\begin{aligned}&\text{(i)}\\&\text{(ii)}\end{aligned}$

Mult. (i) by 3, $9x^2 - 6xy = 45,$

.... (ii) by $2x$ $4x^2 + 6xy = 24x;$

$\therefore$ adding, $13x^2 = 45 + 24x$, or $13x^2 - 24x = 45$, whence $x = 3$ or $-1\frac{2}{13}$.

and from (ii) $x = \frac{1}{3}(12 - 2x) = 2$ or $4\frac{10}{13}$.

Ex. 5. $\left.\begin{aligned} x^2 + y^2 &= 25 \\ 2xy &= 24 \end{aligned}\right\}$ $\begin{aligned}&\text{(i)}\\&\text{(ii)}\end{aligned}$

Here adding, $x^2 + 2xy + y^2 = 49$, whence $x + y = \pm 7$;

subtracting, $x^2 + 2xy + y^2 = 1$, whence $x - y = \pm 1$:

if $\left.\begin{aligned} x + y &= +7 \\ \text{and } x - y &= +1 \end{aligned}\right\}$ $\left.\begin{aligned} x + y &= +7 \\ x - y &= -1 \end{aligned}\right\}$

$\therefore 2x = 8$, and $x = 4$, $2x = 6$, and $x = 3$,

$2y = 6$, and $y = 3$; $2y = 8$, and $y = 4$:

similarly, by combining the equation $x + y = -7$ with each of the two $x - y = \pm 1$, we should get the other two pairs of roots

$$x = -4,\ y = -3, \text{ and } x = -3,\ y = -4.$$

Ex. **61**.

1. $\left.\begin{aligned} \tfrac{1}{10}(3x+5y) + \tfrac{1}{6}(4x-3y) &= 6\tfrac{23}{30} \\ 3x^2 + 2y^2 &= 179 \end{aligned}\right\}$ 2. $\left.\begin{aligned} x^2 + y^2 &= 25 \\ x + y &= 1 \end{aligned}\right\}$ 3. $\left.\begin{aligned} x^2 + y^2 &= 25 \\ 4y + 3x &= 24 \end{aligned}\right\}$

4. $\left.\begin{aligned} 2(x-y) &= 11 \\ xy &= 20 \end{aligned}\right\}$ 5. $\left.\begin{aligned} x^2 + xy &= 66 \\ x^2 - y^2 &= 11 \end{aligned}\right\}$ 6. $\left.\begin{aligned} x - y &= 2 \\ 15(x^2 - y^2) &= 16xy \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x^2}{y^2} &= \frac{85}{9} - \frac{4x}{y} \\ x - y &= 2 \end{aligned}\right\}$ 8. $\left.\begin{aligned} xy &= (x - \tfrac{3}{7})(y + \tfrac{2}{3}) \\ x^2y^2 &= (x^2 + 3)(y^2 - 4) \end{aligned}\right\}$ 9. $\left.\begin{aligned} x + y &= 6 \\ x^3 + y^3 &= 72 \end{aligned}\right\}$

10. $\left.\begin{aligned} 3xy + 2x + y &= 485 \\ 3x &= 2y. \end{aligned}\right\}$ 11. $\left.\begin{aligned} x - y &= 1 \\ x^3 - y^3 &= 19 \end{aligned}\right\}$ 12. $\left.\begin{aligned} x^3 + y^3 &= 189 \\ x^2y + xy^2 &= 180 \end{aligned}\right\}$

13. $\left.\begin{aligned} x + y &= a \\ x^2 + y^2 &= b^2 \end{aligned}\right\}$ 14. $\left.\begin{aligned} xy &= a^2 \\ x - y &= b \end{aligned}\right\}$ 15. $\left.\begin{aligned} \sqrt[3]{x} &= \sqrt[3]{y} = 3 \\ x + y &= 9 \end{aligned}\right\}$ 16. $\left.\begin{aligned} x^2 + xy &= a^2 \\ y^2 + xy &= b^2 \end{aligned}\right\}$

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137. In the solutions of Problems, depending on quadratic and higher equations, there may be two or more values of the root, and these may be *real* quantities, or *impossible.* In the former case, we must consider if any of the roots are excluded by the nature of the question, which may altogether reject *fractional*, or *negative*, or *surd* answers: in the latter case, we conclude that the solution of the proposed question is arithmetically impossible.

Ex. 1. *What number, when added to* 30, *will be less than its square by* 12?

Let x be the number; then $30 + x = x^2 - 12$, whence $x = 7$, or -6: and here the latter root would be excluded, if we require only *positive* numbers.

Ex. 2. *A person bought a number of oxen for* £120; *if he had bought* 3 *more for the same money, he would have paid* £2 *less for each. How many did he buy?*

Let x be the number he bought: then the price actually given for each was $\frac{120}{x}$, and $\therefore \frac{120}{x+3} = \frac{120}{x} - 2$, whence $x = 12$, or -15, which latter root is rejected by the nature of the Problem.

Ex. 3. *The sum of the squares of the digits of a number of two places is* 25, *and the product of the digits is* 12. *Find the number.*

Let x, y be the digits, so that the number will be $10x + y$; then $x^2 + y^2 = 25$, and $xy = 12$, from which equations we get $x = 3$, $y = 4$, or $x = 4$, $y = 3$, and the number will be 34 or 43. In this case both the roots give solutions.

Ex. 4. *Find two numbers such, that their sum, product, and difference of their squares may be all equal.*

Here assume $x + y$ and $x - y$ for the two numbers: [this step should be noticed, as it simplifies much the solution of problems of this kind:] then their sum $= 2x$, their product $= x^2 - y^2$, and the difference of their squares $= 4xy$; $\therefore$ (i) $2x = 4xy$, (ii) $2x = y^2 - y^2$; from (i) $y = \frac{1}{2}$, from (ii) $2x = x^2 - \frac{1}{4}$, whence $x = \frac{1}{2}(2 \pm \sqrt{5})$; and $\therefore x + y = \frac{1}{2}(3 \pm \sqrt{5})$, $x - y = \frac{1}{2}(1 \pm \sqrt{5})$, the numbers required.

Ex. 5. *Find two numbers whose difference is* 10, *and product one-third of the square of their sum.*

Let $x =$ the least, and $x + 10 =$ the greater; then $x(x + 10) = \frac{1}{3}(2x + 10)^2$, whence $x = -5 \pm 5\sqrt{-3}$, which are *impossible*. The question, in fact, amounts to asking for two numbers, x and y, such that $xy = \frac{1}{3}(x + y)^2$, or $3xy = x^2 + 2xy + y^2$, or $xy = x^2 + y^2$, which may be easily shewn to be impossible; for $(x - y)^2$, or $x^2 - 2xy + y^2$, is necessarily positive (being a square quantity), whatever x and y may be, and $\therefore x^2 + y^2$ must be greater than $2xy$.

Ex. 62.

1. There are two numbers, one of which is $\frac{4}{5}$ of the other, and the difference of their squares is 81: find them.

2. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356: find them.

3. There are two numbers, one of which is triple of the other, and and the difference of their squares is 128: find them.

4. In a certain court there are two square grass-plots, a side of one of which is 10 yards longer than a side of the other, and the area of the latter is $\frac{9}{25}$ of that of the former. What are the lengths of the sides?

5. What two numbers make up 14, so that the quotient of the less divided by the greater is $\frac{9}{16}$ of the quotient of the greater divided by the less?

6. A draper bought a piece of silk for £16 4*s*, and the number of shillings which he paid per yard was $\frac{4}{9}$ the number of yards. How much did he buy?

7. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but on the enemy coming in sight, the front was increased by 845 men, and the whole was thus drawn up in 5 lines: find the number of men.

8. What number is that, the sum of whose third and fourth parts is less by 2 than the square of its sixth part?

9. There is a number such that the product of the numbers obtained by adding 3 and 5 to it respectively is less by 1 than the square of its double: find it.

10. There is a rectangular field whose length exceeds its breadth by 16 yards, and it contains 960 square yards: find its dimensions.

11. The difference between the hypothenuse and two sides of a right-angled triangle is 3 and 6 respectively; find the sides.

12. What two numbers are those whose difference is 5, and their sum multiplied by the greater, 228?

13. A labourer dug two trenches, one 6 yards longer than the other, for £17 16*s*, and the digging of each cost as many shillings per yard as there were yards in its length: find the length of each.

14. The plate of a looking-glass is 18 inches by 12, and it is to be framed with a frame of uniform width, whose area is to be equal to that of the glass: find the width of the frame?

15. There are two square buildings, paved with stones, each a foot square. The side of one building exceeds that of the other by 12 feet, and the two pavements together contain 2120 stones: find the sides of the buildings.

16. A person bought a certain number of oxen for £240, and, after losing 3, sold the rest for £8 a head more than they cost him, thus gaining £59 by the bargain: what number did he buy?

17. A tailor bought a piece of cloth for £147, from which he cut off 12 yards for his own use, and sold the remainder for £120 5*s*, charging 5 shillings *per* yard more than he gave for it. Find how many yards there were, and what it cost him *per* yard.

18. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each were increased by 3 feet, the fore-wheel would make only 4 revolutions more than the hind one in the same space. What is the circumference of each?

19. By selling a horse for £24, I lose as much per cent. as it cost me. What was the prime cost of it?

20. Bought two flocks of sheep for £15, in one of which there were 5 more than in the other; each sheep in each flock cost as many shillings as there were sheep in the other flock. How many were there in each?

21. A and B take shares in a concern to the amount altogether of £500: they sell out at *par*, A at the end of 2 years, B of 8, and each receives in capital and profit £297. How much did each embark?

22. A and B distribute £5 each in charity: A relieves 5 persons more than B, and B gives to each 1*s* more than A. How many did they each relieve?

23. There is a number of three digits, of which the last is double of the first: when the number is divided by the sum of the digits, the quotient is 22; and, when by the product of the last two, 11. Find the number.

24. Find three numbers, such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and third, and the third by the sum of the first and second, the products shall be 26, 50, and 56.

We have seen that when we have only one equation between *two* unknowns, the number of solutions is *unlimited*, and the equation is *indeterminate.* We shall here make a few remarks upon the simpler kinds of such equations.

138. If one solution be given of the equation $ax \pm by = c$, all the others may be easily found.

For let $x = \alpha$, $y = \beta$, be one solution of the equation $ax + by = c$; then $ax + by = c = a\alpha + b\beta$, or $a(x - \alpha) + b(y - \beta) = 0$, which equation is satisfied by $x - \alpha = -bt$, $y + \beta = at$, where t may be any quantity whatever, positive or negative. Hence the general values of x and y are given by the expressions $x = \alpha - bt$, $y = \beta + at$.

If the given equation be of the form $ax - by = c$, we should obtain in the same way, $x = \alpha + bt$, $y = \beta + at$, the same as we get by writing $-b$ for b in the above.

If we require only *integral* values of x and y, the n° of solutions will be limited; the above results will still apply, only we must now have α, β, t, all integers.

139. It may be shewn, however, that there can be *no* integral solution of $ax \pm by = c$, if a and b have any common factor not common also to c.

For let $a = md$, $b = nd$, while c does not contain d; then $mdx \pm ndy = c$, or $mx \pm ny = \frac{c}{d} =$ a fraction, which is, of course, impossible for any *integral* values of x and y.

We shall suppose then, in future, that a is *prime* to b.

140. To solve the equation $ax \pm by = c$ in integers. If we can discern one solution, we may apply (138).

Thus $13x - 9y = 17$ is satisfied by $x = 2$, $y = 1$; whence $13x - 9y = 17 = 13 \times 2 - 9 \times 1$, or $13(x - 2) = 9(y - 1)$. which is satisfied by $x - 2 = 9t$, $y - 1 = 13t$, so that the solution is $x = 2 + 9t$, $y = 1 + 13t$, where t may have any integral value.

But the following examples will shew the simplest general method of solving such an equation.

Ex. 1. Find the integral solutions of $3x + 5y = 73$.

Divide by the *lowest* coefficient, and express the improper fractions which may arise as mixed numbers;

then $x + y + \frac{2}{3}y = 24 + \frac{1}{3}$, or $x + y - 24 = \frac{1}{3} - \frac{2}{3}y = \dfrac{1-2y}{3}$.

Now, since $x + y - 24$ is integral, so also is $\dfrac{1-2y}{3}$, and any multiple of it; multiply it, then, by such a number as will make the coeff. of y *div. by the den^r^ with rem^r^* 1, *i.e.*, in this case, mult. it by 2;

then $\dfrac{2-4y}{3}$ or $\dfrac{2-y}{3} - y$ is int., $\therefore \dfrac{2-y}{3}$ is int. $= t$ suppose;

hence $2 - y = 3t$, or $y = 2 - 3t$, and $x = \frac{1}{3}(73 - 5y) = 21 + 5t$.

Thus, if we take $t = 0$, then $x = 21$, $y = 2$;
if $t = 1$, $x = 26$, $y = -1$; if $t = -1$, $x = 16$, $y = 5$; &c.

If we require only *positive* integral values of x and y, then we cannot take t *positively* $> \frac{2}{3}$, nor therefore > 0, or *negatively* $> \frac{21}{5}$, nor therefore > 4; hence the values for t range from -4 to 0 inclusively, and thus there will be only 5 *positive* integral solutions.

N.B. It may be shewn that it is always possible to find such a number for multiplier as we have employed above, which shall be *less* than the denominator: and this is the reason why we divide by the *least* of the two coefficients, in order to have the multiplier as low as possible. But when the denominators are both large, a little ingenuity will save the trouble of searching for such a nnmber, by some such reasoning as that in the next Ex., it being noticed, that the point to be aimed at is, to get the coefficient of y (or of x, as the case may be) in the numerator to be *unity*.

Ex. 2. Solve in positive integers $39x - 56y = 11$.

Here $x - y - \frac{17}{39}y = \frac{11}{39}$; $\therefore \dfrac{17y + 11}{39}$ is int., and $\therefore \dfrac{34y + 22}{39}$,

and $\therefore y - \dfrac{34y + 22}{39}$ or $\dfrac{5y - 22}{39}$, and $\therefore \dfrac{40y - 176}{39}$ or $y - 4 + \dfrac{y - 20}{39}$;

let $\dfrac{y - 20}{39} = t$; $\therefore y = 39t + 20$, and $x = \frac{1}{39}(11 + 56y) = 56t + 29$.

If we take $t = 0$, then $x = 29$, $y = 20$, which is the *least* positive integral values they admit of: but the number of such values is here *unlimited*, since we may take *any* positive value for t.

Ex. 3. Find the least number which when divided by 14 and 5 will leave remainders 1 and 3 respectively.

Let the number required $N=14x+1=5y+3$; then $14x-5y=2$, and here $2x+\frac{4}{5}x-y=\frac{2}{5}$, or $2x-y=\frac{2-4x}{5}$; hence $\frac{2-4x}{5}$ is integral, and $\therefore$ also $\frac{8-16x}{5}$, and $\frac{3-x}{5}$, which put $= t$;

whence $x = 3 - 5t$, and $y = \frac{1}{5}(14x - 2) = 8 - 14t$.

If we take $t = 0$, we have $x = 3$, $y = 8$, which are the *least* positive integral values they admit of, and therefore the least value of N is $14.3 + 1 = 5.8 + 3 = 43$; but the n° of *positive* values is unlimited, since we may take *any* negative value for t.

N.B. It appears from Ex. 1, 2, 3, that when only *positive integral* solutions are required, the n° of them will be *limited* or not, according as the equation is of the form $ax + by = c$, or $ax - by = c$.

Ex. 4. Find the least integer which is divisible by 2, 3, 4, with remainders 1, 2, 3.

Let $N=2x+1 = 3y + 2 = 4z + 3$: then (i) $2x - 3y = 1$, whence, as before, $x=3t-1$, $y=2t-1$; and (ii) $2x-4z=2$, or $3t-2z=2$, whence $t=2t'$, $z=3t'-1$; $\therefore$ $x=6t'-1$, $y=4t'-1$, $z=3t'-1$, whence, putting $t' = 1$, we get $x = 5$, and $N = 2x + 1 = 11$.

Ex. 5. In how many ways may £80 be paid in £s and guineas?

Let $x =$ n° of £s, $x =$ n° of guineas; then $20x + 21y =$ n° of shillings in £80 $= 1600$, and $x+y+\frac{1}{20}y=80$: put $\frac{1}{20}y=t$; $\therefore$ $y=20t$, and $x = \frac{1}{20}(1600 - 21y) = 80 - 21t$, which gives *four* solutions, or rather *three*, if we omit the solution $t = 0$, which gives $y = 0$.

[In the Answers we shall omit all *zero-values* for x or y.]

Ex. 63.

1. Find the positive integral solutions of

$2x + 3y = 9$, $4x + 29y = 150$, $3x + 29y = 151$, $7x + 15y = 225$.

2. Find the least positive integral solution of

$19x - 14y = 11$, $17x = 7y + 1$, $23x - 9y = 929$, $8x = 23y + 19$.

3. Find the number of positive integral solutions of

$3x + 4y = 39$, $8x + 13y = 500$, $7x + 13y = 405$, $2x + 7y = 125$.

4. Given $x - 2y + z = 5$ and $2x + y - z = 7$, find the least values of x, y, z, in positive integers.

5. A person distributed 4*s* 2*d* among some beggars, giving 7*d* each to some, and 1*s* each to the rest: how many were there in all?

6. In how many ways could 12 guineas be made up of half-guineas and half-crowns? In how many ways, of guineas and crowns?

7. How many fractions are there with denominators 12 and 18, whose sum is $\frac{25}{36}$?

8. *A* wishes to pay *B* a debt of £1 12*s*, but has only half-crowns in his pocket, while *B* has only fourpenny-pieces; how may they settle the matter most simply between them?

9. What is the least number which, divided by 3 and 5, leaves remainders 2 and 3 respectively? What is the least, which divided by 3 and 7, leaves remainders 1 and 2.

10 A person buys two pieces of cloth for £15, the one at 8*s*, the other at 11*s* *per* yard, and each containing more than 10 yards: how many yards did he buy altogether?

11. In how many ways can £1 be paid in half-crowns, shillings, and sixpences, the number of coins used at each payment being 18?

12. A person counting a basket of eggs, which he knows are between 50 and 60, finds that when he counts them 3 at a time there are 2 over, but when he counts them 5 at a time, there are 4 over: how many were there in all?

13. If I have 9 half-guineas and 6 half-crowns in my purse, how may I pay a debt of £4 11*s* 6*d*?

14. A person in exchange for a certain number of pieces of foreign gold, valued at 29*s* each, received a certain number of sovereigns under fifty, and 1*s* over: what was the sum he received?

15. A French *louis* contains 20 *francs*, of which 25 make £1: how can I pay at a shop a bill of 45 *fr* most simply, by paying Eng. and receiving Fr. gold only? Shew that I cannot pay a debt of 45*s*.

16. A person bought 40 animals, consisting of calves, pigs, and geese, for £40; the calves cost him £5 a piece, the pigs £1, and the geese a crown: how many did he buy of each?

17. Find the least integer, which, when divided by 7, 8, 9, respectively, shall leave remainders 6, 7, 8.

18. Three chickens and one duck sold for as much as two geese; and one chicken, two ducks, and three geese were sold together for 25*s*: what was the price of each?

19. Find the least odd number, which, when divided by 3, 5, 7, shall leave remainders 2, 4, 6.

20. Find the least multiple of 7, which, divided by 2, 3, 4, 5, 6, leaves always *unity* for remainder.

CHAPTER X.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

141. QUANTITIES are said to be in *Arithmetical Progression*, when they proceed by a common *difference.* Thus, 1, 3, 5, 7, &c., 8, 4, 0, −4, &c., a, $a+d$, $a+2d$, $a+3d$, &c., are in A. P., the common differences being 2, −4, d, respectively, which are found by *subtracting any term from the term following.*

142. *Given* a *the first term, and* d *the common difference of an* AR. *series, to find* l *the* n^th^ *term, and* S *the sum of* n *terms.*

Here the series will be a, $a+d$, $a+2d$, $a+3d$, &c., where the coeff. of d in any term is just *less by one* than the No. of the term: thus in the 2nd term we have d, *i. e.* $1d$, in the 3rd, $2d$, in the 4th, $3d$, &c., and so in the n^{th} term we shall have $(n-1)d$; hence $l = a + (n-1)d$.

Again $S = a+(a+d)+(a+2d)+\text{\&c.}+(l-2d)+(l-d)+l$, and also $S = l + (l-d) + (l-2d) + \text{\&c.} + (a+2d) + (a+d)+a$;

$$\therefore 2S = (a+l)+(a+l)+(a+l)+\text{\&c.} = (a+l)\,n;$$

$$\therefore S = (a+l)\frac{n}{2} = \{2a+(-1)d\}\frac{n}{2}, \text{ since } l = a + (n-1)d.$$

Ex. 1. Find the 10th term and the sum of 10 terms of 1, 5, 9, &c.
Here $a = 1$, $d = 4$, $n = 9$;
$\therefore\ l = 1 + (10-1)\,4 = 1 + 9 \times 4 = 37$; $S = (1+37) \times \frac{10}{2} = 190$.

Ex. 2. Find the 9th term and the sum of 9 terms of 7, $5\frac{1}{2}$, 4, &c.
Here $a = 7$, $d = -\frac{3}{2}$, $n = 9$;
$\therefore\ l = 7 + (9-1) \times -\frac{3}{2} = 7 - 8 \times \frac{3}{2} = -5$; $S = (7-5) \times \frac{9}{2} = 9$.

Ex. 3. Find the 13th term of the series −48, −44, −40, &c.
Here $a = -48$, $d = 4$, $n = 13$;
$\therefore\ l = -48 + (13-1)\,4 = -48 + 12 \times 4 = 0.$

Ex. 4. Find the sum of 7 terms of $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+$ &c.

Here $a=\frac{1}{2}$, $d=-\frac{1}{6}$, $n=7$; and here we are not required to find l; ∴, using the second formula, $S=(1+6\times-\frac{1}{6})\frac{7}{2}=(1-1)\frac{7}{2}=0$.

In this case the series, continued, is $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}$, &c., where the first 7 terms together amount to zero.

Ex. 64.

Find the last term and the sum of

1. $2+4+6+$ &c. to 16 terms.
2. $1+3+5+$ &c. to 20 terms.
3. $3+9+15+$ &c. to 11 terms.
4. $1+8+15+$ &c. to 100 terms.
5. $-5-3-1-$ &c. to 8 terms.
6. $1+\frac{6}{7}+\frac{5}{7}+$ &c. to 15 terms.

Find the sum of

7. $\frac{2}{3}+\frac{7}{15}+\frac{4}{15}+$ &c. to 21 terms.
8. $4-3-10-$ &c. to 10 terms.
9. $\frac{1}{2}+\frac{3}{4}+1+$ &c. to 10 terms.
10. $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-$ &c. to 13 terms.
11. $1+2\frac{2}{3}=4\frac{1}{3}+$ &c. to 20 terms.
12. $\frac{3}{5}-\frac{14}{10}-\frac{51}{15}-$ &c. to 10 terms.

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143. By means of the equations (i) $l=a+(n-1)d$, (ii) $S=(a+l)\frac{n}{2}$, and (iii) $S=\{2a+(n-1)d\}\frac{n}{2}$, when any three of the quantities a, d, l, n, S are given, we may find the others.

We may also employ them to solve many problems in A. P., as in the following examples.

Ex. 1. The first term of an AR. series is 3, the 13th term, 55; find the common difference.

Since $l=55$, $a=3$, $n=13$, we have by (i) $55=3+12d$, and ∴ $d=4\frac{1}{3}$.

Ex. 2. What No. of terms of the series 10, 8, 6, &c., must be taken to make 30? and what No. to make 28?

(1) $S=30$, $a=10$, $d=-2$; ∴ by (iii) $30=\{20-2(n-1)\}\frac{n}{2}$;

and the roots of this quadratic are 5 and 6, either of which satisfies the question, since the *sixth* term of the series is zero:

(2) $S=28$, $a=10$, $d=-2$; and the values of n are 4 and 7, either of which also satisfies the question, since the 5th, 6th, and 7th terms of the series, viz. 2, 0, −2, together = zero.

Ex. 3. How many terms of the series, 3, 5, 7, &c., make up 24?

Here $S=24$, $a=3$, $d=2$; whence $n=4$ or -6, of which the first only is admissible by the conditions of the Question.

Ex. 4. Insert 3 AR. *means* between 6 and 26.

Here we have to find *three* numbers between 6 and 26, so that the *five* may be in A. P. This case then reduces itself to finding d, when $a = 6$, $l = 26$, and $n = 5$; we have then by (i) $26 = 6 + 4d$, whence $d = 5$, and the means required are 11, 16, 21.

Ex. 5. The sum of three numbers in A. P. is 21, and the sum of their squares, 179; find them.

Let $a - d$, a, $a + d$, represent the three numbers (which is often a convenient assumption in problems of this kind);

then $(a - d) + a + (a + d) = 21$, and $(a - d)^2 + a^2 + (a + d)^2 = 179$, from which equations $a = 7$, $d = \pm 4$, and the Nos. are 3, 7, 11.

Ex. 65.

1. The first term of an AR. series is 2, the common difference 7, and the last term 79; find the number of terms.

2. The sum of 15 terms of an arithmetic series is 600, and the common difference is 5; find the first term.

3. The first term is $13\frac{1}{15}$, the common difference $-\frac{2}{3}$, and the last term $\frac{2}{5}$; find the number of terms.

4. The sum of 11 terms is $14\frac{4}{7}$, and the common difference is $\frac{2}{7}$; find the first term.

5. Insert 4 AR. means between 2 and 17, and 4 between 2 and -18.

6. Insert 9 A. M. between 3 and 9, and 7 between -13 and 3.

7. Insert 10 A. M. between -7 and 114, and 8 between -3 and $-\frac{3}{4}$.

8. Insert 9 A.M. between $-2\frac{3}{4}$ and $4\frac{3}{4}$, and 9 between $-3\frac{2}{5}$ and $2\frac{3}{5}$.

9. Find the 3 Nos. in A. P., whose sum shall be 21, and the sum of the first and second $= \frac{3}{4}$ that of the second and third.

10. There are 3 Nos. in A. P., whose sum is 10, and the product of the second and third $33\frac{1}{3}$; find them.

11. Find 3 Nos. whose common difference is 1, such as the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

12. The first term is $n^2 - n + 1$, the common difference 2; find the sum of n terms.

13. How many strokes a-day do the clocks of Venice make, which strike from one to twenty-four?

14. How many strokes does a common clock make in 12 hours? and how many, if it strikes also the half-hours?

15. A debt can be discharged in a year by paying one shilling the first week, three the second, five the third, &c.: required the last payment and the amount of the debt.

16. One hundred stones being placed on the ground at the distance of a yard from one another, how far will a person travel, who shall bring them, one by one, to a basket, placed at the distance of a yard from the first stone?

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144. Quantities are said to be in *Geometrical Progression*, when they proceed by a common *factor*.

Thus 1, 3, 9, &c., 4, 1, $\frac{1}{4}$, &c., $-\frac{1}{3}$, $\frac{4}{15}$, $-\frac{16}{75}$, &c., a, ar, ar^2, &c., are in G. P., the common factors or *ratios* (as they are called) being 3, $\frac{1}{4}$, $-\frac{4}{5}$, r, respectively, which may be found *by dividing any term by the term preceding.*

145. *Given* a *the first term and* r *the common ratio of a* GEOM. *series, to find* l *the* nth *term and* S *the sum of* n *terms.*

Here the series will be a, ar, ar^2, ar^3, &c., where the index of r in any term is just *less by one* than the number of the term: thus, in the 2nd term we have r, i. e., r^1, in the 3rd, r^2, in the 4th, r^3, &c., and so in the n^{th} term we shall have r^{n-1}; hence $l = ar^{n-1}$.

Again $S = a + ar + ar^2 + \&c. + ar^{n-1}$,

and $\therefore$ $rS = ar + ar^2 + ar^3 + \&c. + ar^n$;

$\therefore rS - S = ar^n - a$, the other terms disappearing;

hence $S = \dfrac{ar^n - a}{r-1} = a\,\dfrac{r^n - 1}{r-1}$, or $= \dfrac{rl - a}{r-1}$, since $rl = ar^n$.

Ex. 1. Find the 6th term and the sum of 6 terms of 1, 2, 4, &c.

Here $a = 1$, $r = 2$, $n = 6$;

$$\therefore l = 1 \times 2^{6-1} = 1 \times 2^5 = 1 \times 32 = 32; \text{ and } S = \frac{64-1}{2-1} = 63.$$

Ex. 2 Find the 8th term and the sum of 8 terms of 81, −27, 9, &c.

Here $a = 81$, $r = -\frac{1}{3}$, $n = 8$;

$$\therefore l = 81 \times (-\tfrac{1}{3})^7 = 3^4 \times -\frac{1}{3^7} = -\frac{1}{3^3} = -\frac{1}{27}; \text{ and } S = \frac{\frac{1}{81} - 81}{-\frac{1}{3} - 1} = 60\tfrac{20}{27}$$

Ex. 3. Find the sum of $3 - 6 + 12 -$ &c. to 6 terms.

Here $a = 3$, $r = -2$, $n = 6$; therefore, without finding l,

$$S = 3 . \frac{(-2)^6 - 1}{-2-1} = 3 . \frac{64-1}{-3} = -63.$$

Ex. 4. Find the sum of $1 - \frac{4}{3} + \frac{16}{9} -$ &c. to 4 terms.

Here $a = 1,\ r = -\frac{4}{3},\ n = 4;$

$$\therefore S = 1 \times \frac{\left(-\frac{4}{3}\right)^4 - 1}{-\frac{4}{3} - 1} = \frac{\frac{4^4}{3^4} - 1}{-\frac{4}{3} - 1} = \frac{\frac{4^4 - 3^4}{3^4}}{-\frac{7}{3}} = -\frac{3}{7} \times \frac{256 - 81}{3^4} = -\frac{175}{7.3^3} = -\frac{25}{27}.$$

Ex. 5. Find the sum of $2\frac{1}{2} - 1 + \frac{2}{5} -$ &c. to 5 terms.

Here $a = \frac{5}{2},\ r = -\frac{2}{5},\ n = 5;$

$$\therefore S = \frac{5}{2} \frac{\left(-\frac{2}{5}\right)^5 - 1}{-\frac{2}{5} - 1} = \frac{5}{2} \cdot \frac{-\frac{2^5}{5^5} - 1}{-\frac{2}{5} - 1} = \frac{5}{2} \cdot \frac{\frac{2^5 + 5^5}{5^5}}{\frac{7}{5}}$$

$$= \frac{5}{2} \cdot \frac{5}{7} \cdot \frac{32 + 3125}{5^5} = \frac{3157}{14.5^3} = 1\tfrac{201}{250}.$$

Ex. 66.

Find the last term and the sum of

1. $1 + 4 + 16 +$ &c. to 4 terms.
2. $5 + 20 + 80 +$ &c. to 5 terms.
3. $3 + 6 + 12 +$ &c. to 6 terms.
4. $2 - 4 + 8 -$ &c. to 8 terms.
5. $1 - 4 + 16 -$ &c. to 7 terms.
6. $1 - 2 + 2^2 -$ &c. to 10 terms.

Find the sum of

7. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} +$ &c. to 8 terms.
8. $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} +$ &c. to 6 terms.
9. $\frac{3}{2} + 1 + \frac{2}{3} +$ &c. to 6 terms.
10. $3 - \frac{1}{2} + \frac{1}{12} -$ &c. to 5 terms.
11. $9 - 6 + 4 -$ &c. to 9 terms.
12. $100 - 40 + 16 -$ &c. to 5 terms.

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146. If r be a *proper* fraction, that is, if r be < 1, its powers, r^2, r^3, &c., r^n will, *a fortiori*, be also < 1, and, therefore, ar^n will be $< a$: hence, instead of writing $S = \dfrac{ar^n - a}{r - 1}$, in which fraction both numerator and denominator are *negative*, we may write, in this case,

$$S = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now the *greater* we take the value of n (that is, the more terms we take of the series), the *less* will be the value of ar^n; and, by taking n sufficiently great, we may get ar^n as small as we please, only never so small as actually to *vanish*. If ar^n vanished, we should have the sum of the series $= \dfrac{a}{1-r}$; but since, however small may be the value of ar^n, the second fraction will never actually become *zero*, it follows that the sum of the series will never actually reach the above value, though, by increasing n, that is, taking more terms of the series, it may be made to approach it as nearly as we please.

On this account $\dfrac{a}{1-r}$ is said to be the *Limit* of the sum of the series, $a + ar + ar^2 +$ &c., or sometimes (but less correctly) the sum of the series *ad infinitum.*

It is common to denote the Limit of such a sum by Σ.

Ex. 1. Find the limit of the sum of the series $1 + \frac{1}{2} + \frac{1}{4} +$ &c.

Here $a = 1$, $r = \frac{1}{2}$; $\therefore \Sigma = \dfrac{1}{1 - \frac{1}{2}} = \dfrac{1}{\frac{1}{2}} = 2$; *i. e.* the more terms we take of this series, the more nearly will their sum $= 2$, but will never actually reach it.

Ex. 2. Sum $2\frac{1}{2} - \frac{1}{2} + \frac{1}{10} -$ &c. *ad infinitum.*

Here $a = 2\frac{1}{2}$, $r = -\frac{1}{5}$; $\therefore \Sigma = \dfrac{\frac{5}{2}}{1 - (-\frac{1}{5})} = \dfrac{\frac{5}{2}}{1 + \frac{1}{5}} = \dfrac{\frac{5}{2}}{\frac{6}{5}} = 2\frac{1}{12}$.

Ex. **67.**

Find the Limit of the sum of the following series:

1. $4+2+1+$ &c.	2. $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+$ &c.	3. $\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-$ &c.
4. $\frac{3}{2}-1+\frac{2}{3}-$ &c.	5. $1-\frac{1}{2}+\frac{1}{4}-$ &c.	6. $1-\frac{2}{5}+\frac{4}{25}-$ &c.
7. $\frac{1}{5}+\frac{1}{15}+\frac{1}{45}+$ &c.	8. $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+$ &c.	9. $2-\frac{3}{4}+\frac{9}{32}-$ &c.
10. $2-1\frac{1}{3}+\frac{3}{9}-$ &c.	11. $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+$ &c.	12. $-3\frac{1}{5}+1\frac{3}{5}-\frac{4}{5}+$ &c.

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147. By means of the equations of G. P., we may solve many problems respecting series of this kind. It is not, however, generally easy to find n, when the other quantities are given, because this quantity occurs in the form of an index. The Student may be able to guess at its value in the simple instances we shall here give; but, in other cases, it could only be found by the aid of logarithms.

Ex. 1. Find a GEOM. series, whose 1st term is 2 and 7th term $\frac{1}{32}$.
Here $a=2$, $l=\frac{1}{32}$, $n=7$; $\therefore \frac{1}{32}=2r^6$, and $r^6=\frac{1}{64}$, whence $r=\pm\frac{1}{2}$, and the series is 2, $\pm$ 1, $\frac{1}{2}$, $\pm\frac{1}{4}$, &c.

Ex. 2. Given 6 the second term of a GEOM. series and 54 the fourth, find the first term.

Here $6=ar$, $54=ar^3$; $\therefore \frac{54}{6}=\frac{ar^3}{ar}$, or $9=r^2$; hence $r=\pm 3$, $a=\frac{6}{r}=\pm 2$.

Ex. 3. Insert 3 GEOM. means between 2 and $10\frac{1}{8}$.

Here $\frac{81}{8}$ is the 5th term of a series, whose first term is 2;
$\therefore \frac{81}{8}=2r^4$, and $r^4=\frac{81}{16}$; whence $r=\pm\frac{3}{2}$, and the means are $\pm 3, 4\frac{1}{2}, \pm 6\frac{3}{4}$.

Ex. 68.

1. How many terms of the series 2, $-$ 6, 18, &c. must be taken to make -40?

2. The fifth term of a GEOM. series is 8 times the second, and the third term is 12; find the series.

3. The fifth term of a GEOM. series is 4 times the third, and the sum of the first two is -4; find the series.

4. The population of a country increases annually in G. P., and in 4 years was raised from 10000 to 14641 souls; by what part of itself was it annually increased?

5. The difference between the first and second of 4 numbers in G. P. is 12, and the difference between the third and fourth is 300; find them.

6. Insert 3 G. M. between 2 and 32, and also between $\frac{1}{2}$ and 128.

7. Insert 4 G. M. between $-\frac{1}{10}$ and $3\frac{1}{5}$, and also between $\frac{2}{3}$ and $-5\frac{1}{16}$.

8. The sum of an infinite GEOM. series is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

9. The sum of an infinite GEOM. series is 2, and the second term is $-\frac{3}{2}$; find the series.

10. If $2\frac{1}{4}$, 1, be the first and third terms of a G. P., find the sum of the series *ad infinitum*.

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148. Quantities are said to be in *Harmonical Progression*, when their reciprocals are in A. P.

Thus, since 1, 3, 5, &c., $\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{3}{4}$, &c., are in A. P., their reciprocals 1, $\frac{1}{3}$, $\frac{1}{5}$, &c., 4, $-$ 4, $-\frac{4}{3}$, &c. are in H. P.

The term *Harmonical* is derived from the fact that musical strings of equal thickness and tension will produce harmony when sounded together, if their lengths be as the reciprocals of the AR. series of natural numbers, 1, 2, 3, &c.

We cannot find the sum of *any* No. of terms of an HARM. series; but many problems with respect to such series may be solved by *inverting* the terms, and treating their reciprocals as in A. P.

Ex. 1. Continue to 3 terms each way the series 2, 3, 6.

Since $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ are in A. P. with common difference $-\frac{1}{6}$, the AR. series continued each way is $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6}, -\frac{1}{3}$;

$\therefore$ the HARM. series is $1, \frac{6}{5}, \frac{3}{2}, 2, 3, 6, \infty, -6, -3$.

Ex. 2. Insert 4 HARM. means between 2 and 12.

We must here insert 4 AR. means between $\frac{1}{2}$ and $\frac{1}{12}$, which being $\frac{5}{12}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, hence the HARM. means required are $2\frac{2}{5}$, 3, 4, 6.

Ex. 69.

1. Continue to 3 terms each way, $2, \frac{4}{3}, 1$; $1\frac{1}{2}, 2\frac{1}{7}, 3\frac{3}{4}$; $1, 1\frac{1}{6}, 1\frac{2}{5}$.
2. Insert two H. means between 2 and 4, and six between 3 and $\frac{6}{23}$.
3. Find a fourth HARM. proportional to 6, 8, 12.

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149. *To find* A, G, H *the* AR., GEOM., *and* HARM. *means between* a *and* b.

(i) By (141) $b - A = A - a$; $\therefore 2A = a + b$, and $A = \frac{1}{2}(a + b)$:

(ii) by (144) $\frac{b}{G} = \frac{G}{a}$, $\therefore G^2 = ab$, and $G = \sqrt{ab}$, where, however, unless a and b have the same sign, $\sqrt{ab}$ will be *impossible:*

(iii) by (148) $\frac{1}{b} - \frac{1}{H} = \frac{1}{H} - \frac{1}{a}$; $\therefore aH - ab = ab - bH$, or $H = \frac{2ab}{a+b}$.

150. *To prove that* G *is the* GEOM. *mean between* A *and* H; *and that* A, G, H, *are in order of magnitude,* A *being greatest.*

[We use the sign > for *greater than,* and < for *less than.*]

Since $A = \frac{a+b}{2}$, and $H = \frac{2ab}{a+b}$, $\therefore AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2$;

$\therefore G = \sqrt{AH}$, or G is the GEOM. mean between A and H.

Also $A > H$, if $\frac{a+b}{2} > \frac{2ab}{a+b}$, or if $a^2 + 2ab + b^2 > 4ab$,

or if $a^2 + b^2 > 2ab$; and, this being the case (137),

$\therefore A > H$, and, of course, $> G$, whose value (being the GEOM. mean between them) lies between those of A and H.

151. *Three quantities,* a, b, c, *are in* AR., GEOM., *or* HARM. PROG., *according as*

$$\frac{a-b}{b-c} = \frac{a}{a}, \text{ or } = \frac{a}{b}, \text{ or } = \frac{a}{c}.$$

(i) $\frac{a-b}{b-c} = \frac{a}{a} = 1$; $\therefore a - b = b - c$, and a, b, c, are in A. P.:

(ii) $ab - b^2 = ab - ac$, or $b^2 = ac$; $\therefore \frac{b}{a} = \frac{c}{b}$, and a, b, c, are in G. P.:

(iii) $ac - bc = ab - ac$, or, (dividing each by abc,) $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$, whence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P., and therefore a, b, c, are in H. P.

Ex. 70.

1. Find the AR., GEOM., and HARM. means between 2 and $4\frac{1}{2}$.

2. Find the AR., GEOM., and HARM. means between $3\frac{3}{8}$ and $1\frac{1}{2}$.

3. The sum and difference of the AR. and GEOM. means between two numbers are 9 and 1 respectively; find them.

4. The HARM. mean between two numbers is $\frac{16}{25}$ of the AR., and one of the numbers is 4; find the other.

5. The difference of the AR. and HARM. means between two numbers is $1\frac{4}{5}$; find the numbers, one being four times the other.

6. Find two numbers whose difference is 8, and the HARM. mean between them $1\frac{4}{5}$.

CHAPTER XI.

RATIO, PROPORTION, AND VARIATION.

152. The *Ratio* of one quantity to another is that relation which the former bears to the latter in respect of magnitude, when the comparison is made by considering, not *by how much* the one is greater or less than the other, but *what number of times* it contains it, or is contained in it, *i. e.* what *multiple*, *part*, or *parts*, or, in other words, what *fraction* the first is of the second.

This is, in fact, the way in which we naturally, and, as it were, unconsciously, compare the magnitude of quantities. Thus the mere numerical *difference* between 999 and 1000 is the same as between 1 and 2; but no one would hesitate to say that 999 is much *greater*, compared with 1000 than 1 is, compared with 2. The reason is, that the mind considers intuitively that 999 is a much greater fraction of 1000 than 1 is of 2; and this is what we should express by saying that the ratio of 999 to 1000 is greater than that of 1 to 2. On the other hand, we should say at once that 1001 is much *less*, compared with 1000, than 2 is, compared with 1, the fraction in the former case being less than in the latter.

The ratio, then, of one quantity to another is represented by the fraction obtained by dividing the former by the latter.

Thus, the ratio of 6 to 3 is $\frac{6}{3}$ or 2, that of 15 to 40 is $\frac{15}{40}$ or $\frac{3}{8}$, that of $4a$ to $6b$ is $\frac{4a}{6b}$ or $\frac{2a}{3b}$.

Of course the two quantities compared (if they are not mere numbers, or algebraical quantities expressing numbers) must be of the same kind, or one could not be a fraction of the other.

Thus, the ratio of £9 to £12 is the same as that of 9 cwt. to 12 cwt., or of 9 to 12, or of 3 to 4, or of $\frac{3}{4}$ to 1; since, in each of these pairs of quantities, the first is $\frac{3}{4}$ of the second, and hence $\frac{3}{4}$ is the value of each of these ratios; in saying which we may suppose, if we please, a tacit reference to 1, *i. e.*, in saying that the ratio of £9 to £12 is $\frac{3}{4}$, we may either imply that £9 is $\frac{3}{4}$ of £12, or that the ratio of £9 to £12 is the same as that of $\frac{3}{4}$ to 1.

153. The ratio of one quantity to another is expressed by two points placed between them, as $a : b$ and the former is called the *antecedent* term of the ratio, the latter the *consequent*.

A ratio is said to be a ratio of *greater* or *less* inequality, according as the antecedent is greater or less than the consequent.

The ratio of $a^2 : b^2$ is called the *duplicate* (*i. e.* squared) ratio of $a : b$, $a^3 : b^3$ the triplicate ratio of $a : b$, &c.

154. Problems upon ratios are solved by representing them by their corresponding fractions, which may now be treated by the ordinary rules.

Thus ratios are *compared* with one another, by reducing the corresponding fractions to common den[rs], and comparing the num[rs]; and, if these fractions be multiplied together, the resulting fraction is said to be the *ratio compounded of the ratios* represented by them.

Ex. 1. Compare the ratios 5 : 7 and 4 : 9.

Ans. $\frac{45}{63}$, $\frac{28}{63}$; whence $5 : 7 > 4 : 9$.

Ex. 2. Find the ratio of $\frac{5}{7} : \frac{4}{9}$. *Ans.* $\frac{5}{7} \div \frac{4}{9} = \frac{5}{7} \times \frac{9}{4} = \frac{45}{28}$.

Ex. 3. What is the ratio compounded of 2 : 3, 6 : 7, 14 : 15?

Ans. $\frac{2}{3} \times \frac{6}{7} \times \frac{14}{15} = \frac{8}{15}$ or 8 : 15.

155. A ratio of greater inequality is diminished, and of less inequality increased, by *adding* the same quantity to both its terms.

For $\dfrac{a}{b} \gtrless \dfrac{a+x}{b+x}$, as $ab+ax \gtrless ab+bx$, as $ax \gtrless bx$, as $a \gtrless b$.

In like manner it may be shewn that a ratio of greater inequality is increased, and of less diminished, by *subtracting* the same quantity from both its terms.

Ex. 71.

1. Compare the ratios $3:4$ and $4:5$; $13:14$ and $23:24$; $3:7$, $7:11$, and $11:15$.
2. Of $a+b:a-b$ and $a^2+b^2:a^2-b^2$, which is $>$, supposing $a>b$?
3. Which is less of $x+y:y$ and $4x:x+y$? of $x^2+y^2:x+y$ and $x^3+y^3:x^2+y^2$? of x^2+y^2 and $x^6+y^6:x^4-x^2y+x^2y^3-xy^3+y^4$?
4. Find the ratio compounded of $3:5$, $10:21$, and $14:15$; of $7:9$, $102:105$, and $15:17$.
5. Find the ratio compounded of $\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3}$ and $\frac{a^2-ax+x^2}{a+x}$.
6. Compound $x^2-9x+20:x^2-6x$ and $x^2-13x+42:x^2-5x$.
7. Compound the ratios $a+b:a-b$, $a^2+b^2:(a+b)^2$, $(a^2-b^2)^2:a^4-b^4$.
8. What is the ratio compounded of the duplicate ratio of $a+b:a-b$, and the difference of the duplicate ratios of $a:a$ and $a:b$, supposing $a>b$?
9. What quantity must be added to each term of the ratio $a:b$, that it may be equal to the ratio $c:d$?
10. Shew that $a-b:a+b \gtrless a^2-b^2:a^2+b^2$, according as $a:b$ is a ratio of less or greater inequality?

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156. When two ratios are *equal*, the four quantities composing them are said to be *proportional* to one another: thus, if $a:b=c:d$, *i. e.* if $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d, are proportionals. This is expressed by saying that *a is to b as c is to d*, and denoted thus, $a:b::c:d$.

The first and last quantities in a proportion are called the *Extremes*, the other two the *Means*.

Problems on proportions, like those on ratios, are solved by the use of fractions.

157. *When four quantities are proportionals, the product of the extremes is equal to the product of the means.*

For if $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$.

Hence, if three terms of a proportion are given, we can find the other; thus

$$a = \frac{bc}{d},\ b = \frac{ad}{c},\ c = \frac{ad}{b},\ d = \frac{bc}{a}.$$

COR. If $a : b :: b : c$, then $ac = b^2$.

158. *If the product of two quantities be equal to that of two others, the four are proportionals, those of one product being the extremes, and of the other the means.*

For if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$;

and $\therefore$ $a : b :: c : d$, or $a : c :: b : d$, in which proportions a, d are the extremes, and b, c the means.

So if $ac = b^2$, $a : b :: b : c$.

159. *If 3 quantities are prop^ls, the first has to the third the duplicate ratio of that which it has to the second.*

For if $\frac{a}{b} = \frac{b}{c}$, then $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$;

$\therefore$ $a : c$ is the duplicate ratio of $a : b$ (153).

160. *When four magnitudes are proportionals, if any equimultiples whatever be taken of the first and third, and any whatever of the second and fourth, then, if the multiple of the first be $>$, $=$, $<$ that of the second, the multiple of the third shall be $>$, $=$, $<$ that of the fourth.*

For if $\frac{a}{b} = \frac{c}{d}$, we have $\frac{ma}{nb} = \frac{mc}{nd}$, where m and n may be any quantities whatever; and hence it follows that, if $ma >, =, < nb$, so also is $mc >, =, < nd$.

161. Conversely, *If there be four magnitudes such, that, when any equimultiples whatever of the second and third are taken, and any whatever of the second and fourth, it is found, that if the multiple of the first be $>$, $=$, $<$ that of the second, that of the third is always*

$>, =, <$ *that of the fourth, then these four quantities are proportionals.*

For, let a, b, c, d be such that, any equimultiples, ma, mc, being taken of the first and third, and any nb, nd, of the second and fourth, it is found that according as $ma >, =, < nb$, so also is $mc >, =, < nd$; and let e be the fourth proportional to a, b, c.

Then, since $\frac{a}{b} = \frac{c}{e}$, $\therefore \frac{ma}{nb} = \frac{mc}{ne}$ for all values of m and n; suppose m and n to be taken such that $ma = nb$, then also $mc = ne$: but when $ma = nb$, by our hyp., $mc = nd$; hence $nd = ne$, or $d = e$; and $\therefore \frac{a}{b} = \frac{c}{d}$, or a, b, c, d are proportionals.

162. *If* a : b : : c : d, *and* b : e : : d : f, *then* a : e : : c : f.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{b}{e} = \frac{d}{f}$; $\therefore \frac{a}{b} \times \frac{b}{e} = \frac{c}{d} \times \frac{d}{f}$, or $\frac{a}{e} = \frac{c}{f}$

This is the proposition *ex æquali*, referred to in Euc. v.

163. *If* a : b : : c : d, *and* e : f : : g : h, *then* ae : bf : : cg : dh.

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{e}{f} = \frac{g}{h}; \therefore \frac{ae}{bf} = \frac{cg}{dh}.$$

This is called *compounding* the two proportions, and so we may compound any number of such proportions.

164. If 4 quantities form a proportion, we may derive from them many other proportions, all equally true.

Thus, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{ma}{mb} = \frac{c}{d}$, or $ma : mb : : c : d$;

similarly

$ma : b : : mc : d$, $a : mb : : c : md$, $a : b : : mc : md$;

and, in like manner, $\frac{a}{m} : \frac{b}{m} : : c : d$, $a : \frac{b}{m} : : c : \frac{d}{m}$, &c.;

that is, either the *first* or *fourth* terms of any proportion may be multiplied or divided by any quantity, provided that either the *second* or *third* be multiplied or divided by the same.

Hence we may get rid of fractions, when occurring in proportions, by multiplying the 1st and 2nd, or 1st and 3rd, &c. terms by the L. C. M. of their denrs; thus, if $\frac{1}{9}a : \frac{1}{12}b :: \frac{3}{40} : \frac{2}{25}$, (multiplying 1st and 2nd by 36, 3rd and 4th by 200), we have $4a : 3b :: 15 : 16$.

165. Again, all the results of (85–88) may be applied to proportional quantities.

Thus, if $a : b :: c : d$, then *inv.*, $b : a :: d : c$, or *alt.*, $a : c :: b : d$; so also, $a \pm b : a :: c \pm d : c$, $a \pm b : b :: c \pm d : d$,

$a + b : a - b :: c + d : c - d$, $ma + nb : ma - nb :: mc + nd : mc - nd$, &c. with similar propns, having a^n, b^n, c^n, d^n, in the place of a, b, c, d.

In like manner, if $a : b :: c : d :: e : f ::$ &c., by which it is meant that $a : b :: c : d$, or $a : b :: e : f$, or $c : d :: e : f$, &c., so that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ &c., then we have $a : b :: a + c + e +$ &c. $: b + d + f +$ &c.; that is, *If any quantities be in continued proportion, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.*

So also $a : b :: ma + nc + pe +$ &c. $: mb + nd + pf +$ &c.,

$a^n : b^n :: ma^n + nc^n + pe^n +$ &c. $: mb^n + nd^n + pf^n +$ &c.,

with other similar proportions, which may be proved as in (88).

Ex. 1. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

Since $d = \frac{bc}{a}$, (157) this is $\frac{\frac{1}{3} \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{6}$.

Ex. 2. Find a mean proportional to 2, and 8.

Since $b^2 = ac$, (157) this is $\sqrt{(2 \times 8)} = \sqrt{16} = 4$.

Ex. 3. If $a : b = c : d$, express $(a+d)-(b+c)$ in terms of a, b, c only,

$$\text{Here } (a+d)-(b+c) = \left(a + \frac{bc}{a}\right) - (b+c) = \frac{a^2 - ab - ac + bc}{a} = \frac{(a-b)(a-c)}{a}$$

Ex. 72.

1. Find a fourth proportional to 3, 5, 6; to 12, 5, 10; to $\frac{2}{7}$, $\frac{3}{4}$, $\frac{5}{6}$.
2. Find a third proportional to 4, 6; to 2, 3; to $\frac{5}{3}$, $\frac{5}{7}$.
3. Find a mean proportional to 4, 9; to 4, $\frac{16}{25}$; to $1\frac{7}{9}$, $1\frac{9}{16}$.
4. If $a : b :: b : c$, then $a^2 + b^2 : a + c :: a^2 - b^2 : a - c$.

5. If $\frac{a}{b} = \frac{c}{d}$, shew that $(a + b)(c + d) = \frac{b}{d}(c + d)^2 = \frac{d}{b}(a + b)^2$.

6. If $a : b :: c : d$, and $m : n :: p : q$,
then $ma + nb : ma - nb :: pc + qd : pc - qd$.

7. If $a : b :: b : c$, then $a^2 - b^2 : a :: b^2 - c^2 : c$.

8. If $a : b :: c : d :: e : f$, then $a - e : b - f :: c : d$.

9. If $a : b :: b : c$, then $ma^2 - nb^2 : ma - nc :: pa^2 + qb^2 : pa + qc$.

10. If $a : b :: b : c$, then $a - 2b + c = \frac{(a - b)^2}{a} = \frac{(b - c)^2}{c}$.

11. If $\frac{a}{b} = \frac{c}{d}$, then $\left(\frac{1}{a} + \frac{1}{d}\right) - \left(\frac{1}{b} + \frac{1}{c}\right) = \frac{(a - b)(a - c)}{abc}$.

12. If $a : b = b : c$, then $a + b + c : a - b + c :: (a + b + c)^2 : a^2 + b^2 + c^2$.

13. Solve the equations

(i) $\sqrt{x} + \sqrt{b} : \sqrt{x} - \sqrt{b} :: a : b$. (ii) $x + a : 2x - b :: 3x + b : 4x - a$.

(iii) $\left.\begin{array}{l} x + y + 1 : x + y + 2 :: 6 : 7 \\ y + 2x : y - 2x :: 12x + 6y - 3 : 6y - 12x - 1 \end{array}\right\}$.

(iv) $x : 27 :: y : 9 :. 2 : x - y$.

14. What number is that to which, if 1, 5, and 13 be severally added, the first sum shall be to the second as the second to the third?

15. Find two numbers in the ratio of $2\frac{1}{2} : 2$, such that, when diminished each by 5, they shall be in that of $1\frac{1}{3} : 1$.

16. A railway passenger observes that a train passes him, moving in the opposite direction, in 2″, whereas, if it had been moving in the same direction with him, it would have passed him in 30″: compare the rates of the two trains.

17. *A* and *B* trade with different sums: *A* gains £200, *B* loses £50, and now *A*'s stock : *B*'s :: $2 : \frac{1}{2}$; but, if *A* had gained £100 and *B* lost £85, their stocks would have been as $15 : 3\frac{1}{4}$; find the original stock of each.

18. A hare is 50 leaps before a greyhound, and takes four leaps to his three; but two of the greyhound's leaps are as much as three of the hare's: how many leaps must the greyhound take to catch the hare?

19. Divide £500 among *A*, *B*, *C*, in the proportion of 3, 4, 5, and also in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$; and if *A*'s portion be to *B*'s :: 9 : 8, and to *C*'s :: 6 : 5, shew that the shares of *A*, *B*, *C* are in the proportion of $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$.

20. A quantity of milk is increased by watering in the ratio of 4 : 5, and then three gallons are sold; the rest, being mixed with three quarts of water, is increased in the ratio of 6 : 7; how many gallons of milk were there at first?

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166. The value of any Alg. quantity will, of course, depend on the values we give to the letters it contains.

DEF. When two quantities are such, that their ratio is *constant*, that is, remains the same, whatever values we give to the letters they contain, one of them is said to *vary as* the other.

The sign used to denote variation is $\propto$ (read *varies as*).

Thus, $x^2 + 3x \propto 2x^2 + 6x$, since $\dfrac{x^2 + 3x}{2x^2 + 6x} = \dfrac{1}{2}$, whatever be the value of x.

167. Hence, if $A \propto B$, (where A and B are used to denote, not numerical or *constant*, but algebraical or *variable* quantities, such as admit of different values by giving different values to the letters they contain) then, according to the above definition, the value of the ratio $A : B$ will remain constant, whatever may be the values of the quantities A and B themselves. If then we put m to denote this constant value, we have $\dfrac{A}{B} = m$, or $A = mB$; so that, *when one quantity varies as another, they are connected by a constant multiplier.*

Thus $x^2 + 3x = \frac{1}{2}(2x^2 + 6x)$, from which it follows necessarily that $\dfrac{x^2 + 3x}{2x^2 + 6x} = \dfrac{1}{2}$, for all values of x, or, as above stated, $x^2 + 3x \propto 2x^2 + 6x$.

168. Hence also if $A \propto B$, and a, b, be any pair of values of A and B, then for any *other* values of A and B, we have $A : B = m = a : b$, that is, *when one quantity varies as another, if any two pairs of values be taken of them, the four will be proportionals:* or since $A : a :: B : b$, we may state this by saying that if one of them be

changed from any one value (A) to any other value (a), the other will be changed *in the same proportion* from the value (B) corresponding to the first to the value (b) corresponding to the second.

169. The following are terms used in Variation:

1. If $A = mB$, then A is said to vary *directly* as B;

2. If $A = \frac{m}{B}$, A is said to vary *inversely* as B;

3. If $A = mBC$, then A is said to vary *jointly* as B and C;

4. If $A = m\frac{B}{C}$, then A is said to vary *directly* as B, and *inversely* as C.

170. The following results in Variation are noticeable.

(i) If $A \propto B$ and $B \propto C$, then $A \propto C$.

For let $A = mB$, $B = nC$; then $A = mnC$, and $\therefore A \propto C$, since, m, n, being constant, so also is mn.

So also, if $A \propto B$ and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

(ii) If $A \propto C$ and $B \propto C$, $A \pm B \propto C$, and $\sqrt{(AB)} \propto C$.

For let $A = mC$, $B = nC$;

then $A \pm B = mC \pm nC = (m \pm n)C$, and $\therefore A \pm B \propto C$ and $\sqrt{(AB)} = \sqrt{(mC \times nC)} = \sqrt{(mnC^2)} = \sqrt{(mn)}\, C$, and therefore $\sqrt{(AB)} \propto C$.

(iii) If $A \propto BC$, then $B \propto \frac{A}{C}$, and $C \propto \frac{A}{B}$.

For let $A = mBC$, then $B = \frac{1}{m}.\frac{A}{C}$, or $B \propto \frac{A}{C}$; so $C \propto \frac{A}{B}$.

(iv) If $A \propto B$, and $C \propto D$, then $AC \propto BD$.

For let $A = mB$, $C = nD$; then $AC = mnBD$, or $AC \propto BD$.

(v) If $A \propto B$, then $A^n \propto B^n$.

(vi) If $A \propto B$, and P be any other quantity,

then $AP \propto BP$, and $\frac{A}{P} \propto \frac{B}{P}$.

171. *If* A, B, C, *be variable quantities, depending on one another, and it is observed that, when* C *is kept constant,* $A \propto B$, *and when* B *is kept constant,* $A \propto C$; *then, generally, that is, when all three are allowed to change their values together* $A \propto BC$.

For since $A \propto B$, when C is kept constant, A must be of the form mB, where m is some *constant*, and *may*, therefore, contain the constant C, but not B.

[From this we see that A *must* contain B as a factor, but not B^2, B^3, &c., and *may* contain C.]

Again, since $A \propto C$. when B is kept constant, A must be also of the form nC, where n is some *constant*, and *may*, therefore, contain the constant B, but not C.

[From this we see that A *must* contain C, as a factor, but not C^2, C^3, &c., and *may* contain B, as, in fact, we have already shewn it *does*.]

Upon the whole, then, it appears that A must contain both B and C as factors, but no other powers of B or C, and therefore must be of the form pBC, where p is a constant, containing neither B nor C; hence, since $A = pBC$, we have $A \propto BC$, when all three are allowed to change their values together.

The above result may similarly be proved for any number of quantities, B, C, D, &c.; so that, if any quantity vary separately as each of several others, when the rest are kept constant, it varies as their product, when all are allowed to change their values together.

Ex. 1. If $a \propto b^2c$, and 1, 2, 3, be cotemporaneous values of a, b, c, express a in terms of b and c.

Since $a \propto b^2c$, $\therefore a = mb^2c$, where we have to find m; now, when $b=2$ and $c=3$, a becomes 1; $\therefore 1=12m$, or $m=\frac{1}{12}$, and $\therefore a=\frac{1}{12}b^2c$.

Ex. 2. If y = the sum of two quantities, one of which $\propto x$ and the other $\propto x^2$, and when $x = 1$, $y = 6$, when $x = 2$, $y = 20$; express y in terms of x.

Here $y = mx + nx^2$, where we have to find m and n.

now, by the Question, when $x = 1$, $y = 6$, $\therefore$ (i) $6 = m + n$,

and when $x = 2$, $y = 20$, $\therefore$ (ii) $20 = 2m + 4n$;

from which equations $m = 2$, $n = 4$, and $\therefore y = 2x + 4x^2$.

Ex. 73.

1. If $xy \propto x^2 + y^2$, and 3, 4, be contemporaneous values of x and y, express xy in terms of $x^2 + y^2$.

2. If y = the sum of two quantities, whereof one is constant and the other $\propto x$ *inversely*, and when $x = 2$, $y = 0$, when $x = 3$, $y = 1$, find the value of y, when $x = 6$.

3. If y = the sum of two quantities, whereof one is constant, and the other xy, and when $x = 2$, $y = -2\frac{1}{3}$, when $x = -2$, $y = 1$, express y in terms of x.

4. If y = the sum of three quantities, which vary as x, x^2, x^3 respectively, and when $x = 1, 2, 3$, $y = 6, 22, 54$ respectively, express y in terms of x.

5. If y = the sum of three quantities, of which the first $\propto x^2$, the second $\propto x$, and the third is constant; and when $x = 1, 2, 3$, $y = 6, 11, 18$, respectively, express y in terms of x.

6. Given that $z \propto x + y$, and $y \propto x^2$, and that when $x = \frac{1}{2}$, the values of y and z are 1 and $\frac{1}{4}$, express z in terms of x.

7. If $x \propto \frac{z}{y^2}$ and $z^2 \propto \frac{y}{x}$, shew that $x \propto \frac{1}{y} \propto \frac{1}{z}$.

8. The area of any triangle varies jointly as any side, and the perpendicular let fall upon it from the opposite angle; express the area of the right-angled triangle ABC in terms of the sides AC, BC, containing the right angle, it being found that, when the sum of the two sides is 14 feet and the hypothenuse 10 feet, the area is 24 square feet.

CHAPTER XII.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

172. THE *Variations* of any No. of quantities are the different *arrangements* which can be made of them, taking a certain No. at a time together.

Thus, the Var^{ns} of a, b, c, *two* together, are ab, ba, ac, ca, bc, cb.

When *all* are taken together, the Var^{ns} are called *Permutations:* but this distinction is not always observed, the words Variation and Permutation being used by some as synonymous.

173. *The No. of Var^{ns} of* n *different things, taken* r *together, is* n (n—1) (n—2) . . . . (n—r + 1).

Let there be n different things, a, b, c, d, &c.

The No. of Var^{s} which can be formed of these n things, taken *singly*, is, of course, n.

Now let us remove a; there will then be $n-1$ things, b, c, d, &c., and the Var^{ns} of these taken *singly*, will, (as before) be $n-1$. If then we set a before each of these, there will be $n-1$ Var^{ns} of n things, a, b, c, d, &c. taken *two* and *two* together, in which a stands first; similarly there will be $n-1$ such Var^{ns}, in which b stands first; and so of the rest: therefore, on the whole, there will be $n(n-1)$ Var^{ns} of n things taken *two* and *two* together.

Let us again remove a; there will be $n-1$ things, b, c, d, &c., and the Var^{ns} of these, taken *two* and *two* together, will be $(n-1)$ $(n-2)$ by what precedes; and, by the same course of reasoning, it will appear that, on the whole, there will be n $(n-1)$ $(n-2)$ Var^{ns} of n things taken *three* and *three* together.

Suppose then this law to hold for the No. of Var^{ns} of n things a, b, c, d, &c. taken $r-1$ together, which would be, therefore, $n(n-1)(n-2)\ldots\{n-(r-1)+1\}$, or $n(n-1)(n-2)\ldots(n-r+2)$.

Now remove a; there will then be $n-1$ things b, c, d, &c., and the Var^{ns} of these, taken $r-1$ together, would be found from the preceding result, by writing in it $n-1$ for n, and would, therefore, be

$$(n-1)(n-2)\ldots(n-r+1).$$

If now we set a before each of these, there would be $(n-1)(n-2)\ldots(n-r+1)$ Var^{ns} of n things a, b, c, d, &c. taken r together, in which a stands first; similarly, when b stands first, and so of the rest: therefore, on the whole, there would be $n(n-1)(n-2)\ldots(n-r+1)$ Var^{ns} of n things taken r together.

If then the formula represent correctly the No. of Var^{ns} of n things when taken $r-1$ together, it would also when they are taken r together; but we have shown it to be true when they are taken 1, 2, or 3 together; therefore when taken 4 together; and, therefore, when 5 together, &c., that is, it is generally true for all values we can give to r.

174. Hence denoting by V_1, V_2, V_3, &c. V_r the No. of Var^{ns} of n things taken 1, 2, 3, &c. r together, we have, from the preceding formula,

$$V_1 = n,\ \ V_2 = n(n-1),\ \ V_3 = n(n-1)(n-2), \text{ \&c.}$$
$$V_r = n(n-1)\ldots(n-r+1).$$

Cor. If $r=n$, or *all* the quantities are taken together, then the No. of Perm^{ns} (P) of n things, is

$$n(n-1)(n-2)\ldots(n-n+1) = (n-1)(n-2)\ldots 1;$$

or, reversing the order of the factors,

$$P = 1.2.3\ldots n.$$

175. *The No. of Perm^{ns} of* n *letters, whereof* p *are* a's, q *are* b's, r *are* c's, *&c. is*

$$\frac{1.2.3....n}{1.2.3....p \times 1.2.3....q \times \&c.}$$

For let N be the No. of such Perm^{ns}. Suppose now that in any one of them we change the p a's into *different* letters; then these letters might be arranged (174. Cor.) in $1.2.3....p$ different ways, and so instead of this *one* Perm^{n}, in which p letters would have been a's, we shall now have $1.2.3...p$ *different* Perm^{ns}. The same would be true for *each* of the N Perm^{ns}; hence, if the p a's were changed to *different* letters, we should have altogether $1.2.3....p \times N$ different Perm^{ns} of n letters, whereof still q are b's, r are c's, &c.

So if in these the q b's were changed to different letters, we should have $1.2.3....q \times 1.2.3....p \times N$ different Perm^{ns} of n things, whereof still r would be c's, and so we may go on until *all* the n letters are different; but when this is the case we know (174. Cor.) that their whole number of permutations$=1.2.3....n$; hence $1.2.3....p \times 1.2.3....q \times \&c. \times N = 1.2.3....n$,

$$\text{and } N = \frac{1.2.3....n}{1.2.3....p \times 1.2.3....q \times \&c.}.$$

Ex. 1. How many changes can be rung with 5 bells out of 8? How many with the whole peal?

Here $V_5 = 8.7.6.5.4 = 6720$, $P = 8.7.6.5.4.3.2.1 = 40320$.

Ex. 2. How many *different* words may be made with all the letters of the expression a^3b^2c?

Of these 6 letters, 3 are a's, and 2 b's; $\therefore N = \dfrac{1.2.3.4.5.6}{1.2.3 \times 1.2.} = 60$.

Ex. 3. What No. of things is that, whereof the No. of Var^{ns}, taken 3 together, is 20 times as great as the No. of Var^{ns} of half the same No. of things taken 2 together?

Here, if n denote the No. of things required, we have

$$n(n-1)(n-2) = 20\left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}n - 1\right), \text{ whence } n = 6.$$

Ex. 74.

1. How many changes may be rung with 5 bells out of 6, and how many with the whole peal?

2. In how many different ways may 7 persons seat themselves at table?

3. How many different words may be made of all the letters of the words *division*, *insincere*, *commencement*, *baccalaureus?*

4. How many different words may be made of the letters of the expression $a^4b^3c^2d$?

5. The No. of Var^{ns}, 3 together : the No., 4 together :: 1 : 6; find the No. of things.

6. How many different words may be made of all the letters of the words *mammalia*, *caravansera*, *Oroonoko*, *Mississippi?*

7. The No. of things : the No. of Var^{ns}, 3 together :: 1 : 20; find the No. of things.

8. The No. of Var^{ns} of n things, 3 together : the No. of Var^{ns} of $n+2$ things, 3 together, :: 5 : 12; find n.

9. The No. of Var^{ns} of n things, 4 together : the No. of Var^{ns} of $\frac{2}{3}n$ things, 4 together :: 13 : 2; find n.

10. If the No. of Var^{ns} of n things, 3 together, be 12 times as great as the No. of Var^{ns} of $\frac{1}{2}n$ things, 3 together, what is the No. of Perm^{ns} of the same n things?

11. Of what No. of things are the Perm^{ns} 720?

12. There are 7 letters, of which a certain No. are a's; and 210 different words can be made of them; how many a's are there?

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176. The *Combinations* of any No. of quantities are the different sets that can be made of them, taking a certain No. together, without regard to the order in which they are placed.

Thus, the Comb^{ns} of a, b, c, d, 3 together, are *abc*, *abd*, *acd*, *bcd*.

It is readily seen that each *Comb*^{n} will supply as many corresponding *Var*^{ns}, as the No. of quantities it contains admits of Perm^{ns}.

Thus, the Comb^{n} *abc* supplies the 1.2.3 or 6 Var^{ns} *abc*, *acb*, *bac*, *bca*, *cab*, *cba*.

177. *The No. of Comb^{ns} of* n *different things, taken* r *together, is*

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}.$$

For (176) each *Comb^n* of r things will supply $1.2.3\ldots r$ *Var^{ns}* of r things; hence, if C_r denote the No. of Comb^{ns} of n things, r together, we have

$$1.2.3\ldots r\times C_r = \text{No. of Var}^{ns}\text{ of } n \text{ things, } r \text{ together}$$
$$= V_r = n(n-1)(n-2)\ldots(n-r+1);$$
$$\therefore C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}.$$

Cor. Hence $C_1=\frac{n}{1}$, $C_2=\frac{n(n-1)}{1.2}$, $C_3=\frac{n(n-1)(n-2)}{1.2.3}$, &c.

Now it will be seen hereafter that these are the same as the coefficients of the binomial $(1+x)^n$, so that

$$(1+x)^n = 1 + C_1x + C_2x^2 + \&c. + C_nx^n.$$

Hence, putting $x=1$, we have $2^n=1+C_1+C_2+\&c.+C_n$; or the sum of all the Comb^{ns} that can be made of n things, taken 1, 2, 3, &c. n together $= 2^n-1$.

178. The expression for C_r, (by multiplying both num^r and den^r by $1.2.3\ldots(n-r)$) may be put into the form

$$\frac{n(n-1)(n-2)\ldots(n-r+1)\times(n-r)\ldots3.2.1}{1.2.3\ldots r \qquad \times 1.2.3\ldots(n-r)}$$
$$= \frac{1.2.3\ldots n}{1.2.3\ldots r\times 1.2.3\ldots(n-r)} = \frac{\lfloor n}{\lfloor r\lfloor n-r}$$

if we use $\lfloor n$ to denote the continued product $1.2.3\ldots n$.

Hence, writing $n-r$ for r, we have

$$C_{n-r} = \frac{\lfloor n}{\lfloor n-r\lfloor r} = \frac{\lfloor n}{\lfloor r\lfloor n-r} = C_r;$$

or the No. of Comb^{ns} of n things taken $n-r$ together = the No. of them taken r together.

The Comb^{ns} of one of these sets are said to be *Supplementary* to those of the other.

Ex. 1. Find the No. of Comb^{ns} of 10 things, 3 and 6 together?

Here $C_3 = \dfrac{10.9.8}{1.2.3} = 120$, and $C_6 = C_4 = \dfrac{10.9.8.7}{1.2.3.4} = 210$.

Ex. 2. How many words of 6 letters might be made out of the first 10 letters of the alphabet, with two vowels in each word?

In these 10 letters, there are 7 consonants and 3 vowels; and in each of the required words there are to be 4 consonants and 2 vowels: now the 7 consonants can be combined four together in 35 ways, and the 3 vowels, two together, in 3 ways; hence there can be formed $35 \times 3 = 105$ different sets of 6 letters, of which 4 are consonants and 2 vowels: but each of these sets of 6 letters may be *permuted* $6.5.4.3.2.1 = 720$ ways, each of these forming a different *word*, though the whole 720 are composed of the same 6 letters: hence the No. required $= 105 \times 720 = 75600$.

Ex. 75.

1. How many Comb^{ns} can be made of 9 things, 4 together? how many, 6 together? how many, 7 together?

2. How many Comb^{ns} can be made of 11 things, 4 together? how many, 7 together? how many, 10 together?

3. A person having 15 friends, on how many days might he invite a different party of 10? or of 12?

4. How often might a common die be thrown, so as to expose five *different* faces?

5. Find the whole No. of Comb^{ns} of 6 things, 1, 2, &c., 6 together?

6. Four persons are chosen by lot out of 10: in how many ways can this be done? and how often would any one person be chosen?

7. How often may a *different* guard be posted of 6 men out of 60? on how many of these occasions would any given man be taken?

8. The No. of Comb^{ns} of $\frac{1}{3}n$ things, 2 together, is 15; find n.

9. The No. of Comb^{ns} of n things, 3 together, is $\frac{5}{18}$ of the No., 5 together; find n.

10. The No. of Comb^{ns} of $n + 1$ things, 4 together, is 9 times the No. of Comb^{ns} of n things, 2 together; find n.

11. The No. of Comb^{ns} of $\frac{1}{2}$ things, 4 together, is $3\frac{3}{4}$ of the No. of Comb^{ns} of $\frac{1}{3}n$ things, 3 together; find n.

12. How many words of 6 letters may be made out of the 26 letters of the alphabet, with 2 out of the 5 vowels in every word?

CHAPTER XIII.

THE BINOMIAL THEOREM.

179. THE *Binomial Theorem* is a formula, discovered by *Sir Isaac Newton*, by means of which any binomial may be raised to any given power, without going through the ordinary process of Involution. It may be stated as follows: Whatever be the value of n, positive or negative, fractional or integral,

$$(a+x)^n = a^n + \frac{n}{1} a^{n-1}x + \frac{n(n-1)}{1.2} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3} a^{n-3}x^3 + \&c.;$$

where the coefficient of $a^{n-r}x^r = \dfrac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}$,

and this, being the coefficient of the $(r+1)^{\text{th}}$ term of the expansion, where r may represent *any* positive integer whatever, is called the coefficient of the *general* term.

It will be noticed that the coeff^s of x, x^2, &c. x^r, in the above, when n is a *positive integer*, are no other than the Nos. of Combinations of n things taken 1, 2, &c. r, together. On this account we will use the letters C_1, C_2, &c. C_r, to denote these coeff^s in all cases; and so we may write the formula

$$(a+x)^n = a^n + C_1 a^{n-1}x + C_2 a^{n-2}x^2 + \&c. + C_r a^{n-r}x^r + \&c.$$

In this expression, a and x may stand for any quantities whatever; so that

$$(a-x)^n = \{a+(-x)\}^n = a^n + C_1 a^{n-1}(-x) + C_2 a^{n-2}(-x^2 + \&c.$$
$$= a^n - C_1 a^{n-1}x \quad + C_2 a^{n-2}x^2 \quad - \&c.,$$

where the terms are *alternately* positive and negative: and $(1 \pm x)^n = 1 \pm C_1 x + C_2 x^2 \pm C_3 x^3 + C_4 x^4 \pm \&c.$

180. *To prove the Binomial Theorem when the index is a* POSITIVE INTEGER.

We shall find, by actual multiplication, that

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

$$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$$

Assume this Law of Formation to hold for $n - 1$ factors, so that

$$(x + a_1)(x + a_2)\dots(x + a_{n-1}) = x^{n-1} + p_1x^{n-2} + p_2x^{n-3} + \&c. + p_{n-1},$$

where $p_1 = a_1 + a_2 + a_3 + \&c.$, $p_2 = a_1a_2 + a_1a_3 + a_2a_3$ &c., &c. = &c.

$$p_{n-1} = a_1a_2a_3 \dots a_{n-1},$$

then, multiplying by another factor, $x + a_n$, we have

$$\begin{aligned}(x+a_1)(x+a_2)\dots(x+a_n) &= x^n + p_1x^{n-1} + p_2x^{n-2} + \&c. + p_{n-1}x \\ &\quad + a_nx^{n-1} + p^1a_nx^{n-2} + \&c. + p_{n-2}a_nx + p_{n-1}a_n \\ &= x^n + q_1x^{n-1} + q_2x^{n-2} + \&c. + q_{n-1}x + q^n\end{aligned}$$

where $q_1 = p_1 + a_n = a_1 + a_2 + a_3 + \&c. + a_n,$

$$q_2 = p_2 + p_1a_n = a_1a_2 + a_1a_3 + a_2a_3 + \&c. + a_1a_n + a_2a_n + \&c.$$

&c. = &c.

$$q_n = p_{n-1}a_n = a_1a_2a_3 \dots a_{n-1}a_n,$$

that is, if the Law holds for the product of $n - 1$ factors, it holds also for that of n factors: but we have seen above that it *does* hold for *three* factors, therefore for *four*, and therefore for *five*, and so on; that is, it holds generally, when n is a positive integer.

Now, it is easily seen that the terms in q_1, q^2, q_3, &c., are the different Comb^ns^ of the n letters a_1, a_2, a_3, &c. a_n, taken *one*, *two*, *three*, &c. together; and, consequently, the No. of terms in q_1 is C_1, in q_2 is C_2, &c., as in (177). Let us put a for each of a^1, a_2, &c.: then the first side becomes $(x + a)^n$, and *each* of the terms in q_1, q_2, q_3, &c. becomes a, a^2, a^3, &c. respectively; and therefore we have

$$(x + a)^n = x^n + C_1ax^{n-1} + C_2a^2x^{n-2} + \&c.$$

$$= x^n + \frac{n}{1}ax^{n-1} + \frac{n(n-1}{1.2}a^2x^{n-2} + \&c.$$

And, of course, it will follow in like manner, that

$$(a + x)^n = a^n + C_1xa^{n-1} + C_2x^2a^{n-2} + \&c.$$

$$= a^n + C_1a^{n-1}x + C_2a^{n-2}x^2 + \&c.$$

181. *There are only $n+1$ terms in the expansion of $(1+x)^n$, when the index is a positive integer.*

Since the coeff. of the $(r+1)^{\text{th}}$ term $= C_r$, we see that if r be such that the last factor of the num^r, $n-r+1=0$, then the $(r+1)^{\text{th}}$ and all the *following* terms (all of which would involve this factor) will vanish, *i. e.*, the series will have ended with the r^{th} term. Now if $n-r+1=0$, then $r=n+1$; and the series will have ended with the $(n+1)^{\text{th}}$ term.

182. *In the expansion of $(1+x)^n$, the coeff^s of terms, equally distant from the beginning and end, are the* SAME, *when the index is a positive integer.*

The $(r+1)^{\text{th}}$ term from the end (having r *after* it) will be the $\{(n+1)-r\}^{\text{th}}$ or $(n-r+1)^{\text{th}}$ from the beginning, and its coeff. will therefore be C_{n-r}; but, (178)

$$C_r = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}} = C_{n-r},$$

or coeff. of $(r+1)^{\text{th}}$ term from beginning = coeff. of $(r+1)^{\text{th}}$ term from the end.

N. B. The number of terms, $n+1$, being *odd* or *even* as n is *even* or *odd*, it follows that, if n be even, there will be *one* middle term, but if odd, *two* middle terms, which, by (182), will have *equal* coeff^s, and on each side of which the same coeff^s will occur in order. When, therefore, in expanding a binomial with a *positive integral* index, we have passed the middle term or terms, we shall find all the coeff^s repeating themselves; and, instead of calculating those of the remaining terms, we may write down, in inverted order, the coeff^s already found, as in the following examples.

Ex. 1. $(1+x)^4 = 1 + \frac{4}{1}x + \frac{4.3}{1.2}x^2 + \&c. = 1 + 4x + 6x^2 + 4x^3 + x^4.$

We shall not, however, give any more examples of the 3$^{\text{rd}}$, 4$^{\text{th}}$, and 5$^{\text{th}}$ powers of a binomial, which the Student should be able to write down as in (42).

Ex. 2. $(1-x)^7 = 1 - \frac{7}{1}x + \frac{7.6}{1.2}x^2 - \frac{7.6.5}{1.2.3}x^3 + \&c.$

$= 1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7.$

Ex. 3. $(3x-\frac{1}{2}y)^6=(3x)^6-\frac{6}{1}(3x)^5(\frac{1}{2}y)+\frac{6.5}{1.2}(3x)^4(\frac{1}{2}y)^2-\frac{6.5.4}{1.2.3}(3x)^3(\frac{1}{2}y)^3+\&c.$

$=729x^6-6\times243x^5\times\frac{1}{2}y+15\times81x^4\times\frac{1}{4}y^2-20\times27x^3\times\frac{1}{8}y^3$
$+15\times9x^2\times\frac{1}{16}y^4-6\times3x\times\frac{1}{32}y^5+\frac{1}{64}y^6$

$=729x^6-729x^5y+\frac{1215}{4}x^4y^2-\frac{135}{2}x^3y^3+\frac{135}{16}x^2y^4$
$-\frac{9}{16}xy^5+\frac{1}{64}y^6.$

Ex. 76.

1. $(1+x)^{11}$. 2. $(a+x)^7$. 3. $(1-x)^8$. 4. $(a-x)^9$.
5. $(1+x)^{12}$. 6. $(1-2x)^{10}$. 7. $(a-3x)^6$. 8. $(2x+a)^8$.
9. $(2a-3x)^7$. 10. $(1-\frac{1}{2}x)^{10}$. 11. $(1-\frac{1}{3}x)^{11}$. 12. $(\frac{1}{2}x-\frac{1}{3}y)^{12}$.

---

183. *To prove the Binomial Theorem, when the index is* FRACTIONAL *or* NEGATIVE.

It will be sufficient if we can prove the Theorem for the expansion of $(1+x)^n$, that is, if we can shew that for *all* values of n, $(1+x)^n = 1+C_1x+C_2x^2+\&c.$

For then, since $a+x = a\left(1+\frac{x}{a}\right)$, we shall have

$$(a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$$

$$= a^n\left\{1+C_1\left(\frac{x}{a}\right)+C_2\left(\frac{x}{a}\right)^2+\&c.\right\}$$

$$=a^n+C_1a^{n-1}x+C_2a^{n-2}x^2+\&c., \text{ as required.}$$

Let then the series $1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\&c.$, *whatever* be the value of m, be denoted by the symbol $f(m)$. Now, when m is a positive integer, we know that this series represents the expansion of $(1+x)^m$, that is, $f(m) = (1+x)^m$, when m is a positive integer. We shall now shew that this is the case for *all* values of m.

Since $\quad f(m) = 1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \&c.$

$$\therefore f(n) = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \&c.$$

$$\begin{aligned} \text{and } f(m) \times f(n) = 1 + \frac{m}{1}x &+ \frac{m(m-1)}{1.2}x^2 + \&c. \\ + \frac{n}{1}x &+ \quad mn \quad x^2 + \&c. \\ &+ \frac{n(n-1)}{1.2}x^2 + \&c. \\ = 1 + ax &+ \quad bx^2 \quad + \&c. \end{aligned}$$

where we use a, b, &c. to denote the coeff[s], found by addition, of x, x^2, &c., so that

$$a = m + n,\ b = \frac{m(m-1)}{1.2} + mn + \frac{n(n-1)}{1.2},\ \&c.$$

Now b, c, &c. might be reduced to much simpler forms than these, but the process would be tedious: we may find them, however, immediately by the following consideration. Since the above multiplication does not at all depend upon the actual values of m and n, we should still have, by the addition, the same *values* as above for a, b, &c., whether m and n stand for positive or negative, integral or fractional, quantities.

But when m and n are *positive integers*, we know that

$$f(m) = (1+x)^m,\ f(n) = (1+x)^n,$$

and $\therefore f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n}$; and since $m + n$ is here a positive integer, we know also that

$$(1+x)^{m+n} = 1 + \frac{m+n}{1}x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \&c.$$

Here, therefore, we have the values of a, b, &c., when m and n are positive integers: hence also they will be the same, *whatever* be the values of m and n, and we have, therefore, in all cases,

$$f(m)\times f(n)=1+\frac{m+n}{1}x+\frac{(m+n)(m+n-1)}{1.2}x^2+\&c.;$$

or, since this series would be denoted by $f(m+n)$, we have $f(m)\times f(n)=f(m+n)$, for all values of m and n.

The student may easily satisfy himself that the values just obtained for a, b, c, &c. are identical with the former, though simplified in form; thus,

$$b=\frac{m(m-1)}{1.2}+mn+\frac{n(n-1)}{1.2}=\frac{m(m-1)+2mn+n(n-1)}{1.2}$$

$$=\frac{m(m-1+n)+n(m+n-1)}{1.2}=\frac{(m+n)(m+n-1)}{1.2}.$$

Hence $f(m)\times f(n)\times f(p)=f(m+n)\times f(p)=f(m+n+p)$, and similarly for any No. of such factors; *i. e.* the product of any two, or more, such series, as that denoted by $f(m)$, produces another series of precisely the *same form*.

Now, (i), let there be n factors, each $=f\left(\frac{m}{n}\right)$, where m and n are positive integers; then

$$f\left(\frac{m}{n}\right)\times f\left(\frac{m}{n}\right)\times\&c.\ n\text{ factors}=f\left(\frac{m}{n}+\frac{m}{n}+\&c.\ n\text{ terms}\right),$$

$$\text{or }\left\{f\left(\frac{m}{n}\right)\right\}^n=f\left(\frac{mn}{n}\right)=f(m)=(1+x)^m,$$

since m is a positive integer;

$\therefore$ taking the n^{th} root on both sides, $(1+x)^{\frac{m}{n}}=f\left(\frac{m}{n}\right)$.

Hence $f(m)$ is the series for $(1+x)^m$, so long as the index is *positive*, whether it be *integral* or *fractional.*

Again, (ii), let $n=-m$, where m is *positive*, but may be integral or fractional; then

$$f(m)\times f(-m)=f(m-m)=f(0)=1,$$

(since the series becomes $= 1$, if we put 0 for m in it);

$$\therefore f(-m) = \frac{1}{f(m)} = \frac{1}{(1+x)^m}, \text{ since } m \text{ is } \textit{positive},$$

$$= (1+x)^{-m}, \text{ by the Theory of Indices.}$$

Hence $f(-m)$ is the series for $(1+x)^{-m}$, where the index is *negative*, and may be either *integral* or *fractional*.

It follows then that for *all* values of the index, we have

$$(1+n)^n = f(n) = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \&c.$$

184. We have seen (181) that, when the index is a positive integer, this series will stop after $n+1$ terms; when fractional or negative, it will never terminate, but consist of an infinite number of terms, since we cannot then find any value of r, which will make $n-r+1=0$.

Ex. 1.

$$(1+x)^{-2} = 1 + \frac{-2}{1}x + \frac{-2(-2-1)}{1.2}x^2 + \frac{-2(-2-1)(-2-2)}{1.2.3}x^3 + \&c.$$

$$= 1 - \frac{2}{1}x + \frac{2.3}{1.2}x^2 - \frac{2.3.4}{1.2.3}x^3 + \&c. = 1 - 2x + 3x^2 - 4x^3 + \&c.$$

In this Ex there is some trouble in simplyfying coeff^s, and getting rid of superfluous signs: to save this, it will be useful to remember the result of the following *general* example.

Ex. 2.

$$(1 \pm x)^{-n} = 1 \pm \frac{-n}{1}x + \frac{-n(-n-1)}{1.2}x^2 \pm \frac{-n(-n-1)(-n-2)}{1.2.3}x^3 + \&c.$$

$$= 1 \pm \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 \pm \frac{n(n+1)(n+2)}{1.2.3}x^3 + \&c. \ldots \text{(i).}$$

Ex. 3. $(1+x)^{-3} = 1 - \frac{3}{1}x + \frac{3.4}{1.2}x^2 - \frac{3.4.5}{1.2.3}x^3 + \&c.$

$$= 1 - 3x + 6x^2 - 10x^3 + \&c.$$

Ex. 4. $(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}x^3 + \&c.$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{1.2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1.2.3}x^3 + \&c.$$

$$= 1 + \frac{1}{2}x - \frac{1}{1.2.2^2}x^2 + \frac{3}{1.2.3.2^3}x^3 - \&c. = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \&c.$$

Here also it will be well to notice the following general results.

Ex. 5.

$$(1 \pm x)^{\frac{p}{q}} = 1 \pm \frac{\frac{p}{q}}{1}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1.2}x^2 \pm \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{1.2.3}x^3 + \&c.$$

$$= 1 \pm \frac{p}{q}x + \frac{\frac{p}{q}.\frac{p-q}{q}}{1.2}x^2 \pm \frac{\frac{p}{q}.\frac{p-q}{q}.\frac{p-2q}{q}}{1.2.3.}x^3 + \&c.$$

$$= 1 \pm \frac{p}{q}x + \frac{p(p-q)}{1.2.q^2}x^2 \pm \frac{p(p-q)(p-2q)}{1.2.3.q^3}x^3 + \&c. ...(ii).$$

So also

$$(1 \pm x)^{-\frac{p}{q}} = 1 \pm \frac{p}{q}x + \frac{p(p+q)}{1.2.q^2}x^2 \pm \frac{p(p+q)(p+2q)}{1.2.3.q^3}x^3 + \&c. \ldots . (iii).$$

Ex. 6. $(1+x)^{-\frac{2}{3}} = 1 + \frac{2}{3}x + \frac{2.5}{1.2.3^2}x^2 + \frac{2.5.8}{1.2.3.3^3}x^3 + \&c.$

$$= 1 + \tfrac{2}{3}x + \tfrac{5}{9}x^2 + \tfrac{40}{81}x^3 + \&c.$$

Ex. 77.

1. $(1+x)^{-1}$.
2. $(1-3x)^{-1}$.
3. $(1+3x)^{-2}$.
4. $(1-2x)^{-3}$.
5. $(1-\frac{1}{2}x)^{-2}$.
6. $(1+\frac{1}{3}x)^{-3}$.
7. $(1+2x)^{\frac{1}{2}}$.
8. $(1-3x)^{\frac{2}{3}}$.
9. $(1-x)^{-\frac{3}{2}}$.
10. $(1-x^2)^{\frac{7}{3}}$
11. $\frac{1}{\sqrt[3]{1-x}}$.
12. $\frac{1}{\sqrt{1+x^2}}$.

Ex. 7. $(a+2x)^{-4} = a^{-4}\left(1+\frac{2x}{a}\right)^{-4} = a^{-4}\left\{1-\frac{4}{1}\left(\frac{2x}{a}\right)+\frac{4.5}{1.2}\left(\frac{2x}{a}\right)^2 - \&c.\right\}$

$$= a^{-4} - 8a^{-5}x + 40a^{-6}x^2 - 160a^{-7}x^3 + \&c.$$

Ex. 8.

$$(a-2x)^{-\frac{3}{2}} = a^{-\frac{3}{2}}\left(1-\frac{2x}{a}\right)^{-\frac{3}{2}} = a^{-\frac{3}{2}}\left\{1+\frac{3}{2}\left(\frac{2x}{a}\right)+\frac{15}{8}\left(\frac{2x}{a}\right)^2 + \&c.\right\}$$

$$= a^{-\frac{3}{2}}\left\{1 + 3\frac{x}{a} + \frac{15}{2}\frac{x^2}{a^2} + \frac{35}{2}\frac{x^3}{a^3} + \&c.\right.$$

$$= a^{-\frac{3}{2}} + 3a^{-\frac{5}{2}}x + \tfrac{15}{2}a^{-\frac{7}{2}}x^2 + \tfrac{35}{2}a^{-\frac{9}{2}}x^3 + \&c.$$

Ex. 78.

1 $(2-x)^{-2}$.
2. $(3-2x)^{-3}$.
3. $(a+bx)^{-1}$.
4. $(a-b^2x)^{-2}$.
5. $(a^{-\frac{1}{3}}-b^{\frac{1}{3}})^{-6}$.
6. $(a^2-x^2)^{\frac{1}{5}}$.
7. $(a^{-\frac{1}{3}}+b^{-\frac{1}{3}})^{-3}$.
8. $(a-x)^{\frac{1}{3}}$.
9. $(a^5-x^5)^{-\frac{1}{5}}$.
10. $(a^2-x^2)^{\frac{4}{3}}$.
11. $(a^2-x^2)^{-\frac{3}{4}}$.
12. $(ax-x^2)^{-\frac{1}{3}}$

CHAPTER XIV.

NOTATION, DECIMALS, INTEREST, &C.

185. *Notation* is the method of expressing numbers by means of a series of powers of some one fixed number, which is said to be the *radix* or *base* of the *scale*, in which the different numbers are expressed.

Thus in common Arithmetic, all Nos. are expressed in a scale whose base is 10; for 3578 denotes 3000 + 500 + 70 + 8, *i. e.* $3.10^3 + 5.10^2 + 7.10 + 8$; so also 376, when expressed in a scale whose radix is 12, is 274, since $2.12^2+7.12+4=288+84+4=376$.

186. *If* r *be any integer, any No.* N *may be expressed in the form* $N=p_n r^n+p_{n-1}r^{n-1}+\&c.+p_2r^2+p_1r+p_0$, *where the coefficients* p_n, p_{n-1}, &c. *are integers all less than* r.

For divide N by the greatest power of r it contains, suppose r^n; and let the quotient be p_n (which will, of course, be $<r$), and the remainder N_1: then $N=pr^n+N_1$. Similarly $N_1=p_{n-1}r^{n-1}+N_2$, $N_2=p_{n-2}r^{n-2}+N_3$, &c., and thus continuing the process until the rem^r becomes $<r$, p_0 suppose, we have $N=p_nr^n+p_{n-1}r^{n-1}+$ &c. $+p_2r^2+p_1r+p_0$.

Some of the coefficients p_0, p_1, p_2, &c. may vanish, but none can be $> r$. Their values then may range from 0 to $r-1$, and these different values are called the *digits* of the corresponding scale. Hence, including *zero*, there will be r digits in the scale of r.

Thus in the scale of 12, the digits will be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t* and *e*, where *t* and *e* are used to denote the digits 10 and 11.

187. In the *Binary* scale, the radix is 2; in the *Ternary*, 3; in the *Quaternary*, 4; in the *Quinary*, 5; in the *Senary*, 6, &c.; in the *Denary* or *Decimal*, 10;

in the *Undenary*, 11; *in the Duodenary* or *Duodecimal*, 12; &c.

All Nos. are supposed to be expressed in the common or denary scale, unless the contrary is mentioned.

188. *To express any proposed No. in a given scale.*

Let N be the given No. which is to be expressed in the scale of r, in the form $N = p_n r^n + \&c. + p_2 r^2 + p\ r + p_0$: we are to shew how the digits p_n, p_{n-1}, &c. may be found.

Divide N by r; then we shall have

$$\frac{N}{r} = p_n r^{n-1} + \&c. + p_2 r + p_1 + \frac{p_0}{r},$$

i. e. we shall have an integral quotient, $p_n r^{n-1} + \&c. + p_1$ $(= N_1$, suppose,) with remainder p_0; hence the remainder, upon dividing N by r, is p_0, the *last* of the digits.

Again, divide N_1 by r; then we shall have

$$\frac{N_1}{r} = p_n r^{n-2} + \&c, + p_2 + \frac{p_1}{r} = N_2 + \frac{p_1}{r};$$

hence the rem^r, upon dividing N_1 by r is p_1, the *last but one* of the digits; and so dividing N_2 by r, we get p_2, &c.

Ex. Express the common number 3700 in the *quinary*, and convert 37704 from the *nonary* to the *octenary* scale.

Ex. 1.	Ex. 2.
5) 2700	8) 37704
5) 740 ... 0	8) 4311 ... 5
5) 148 ... 0	8) 480 ... 1
5) 28 ... 3	8) 54 ... 4
5) 5 ... 4	6 ... 1
Ans. 104300. 1 ... 0	*Ans.* 61415.

Notice that in Ex. 2, the radix is 9; and therefore, when, in beginning the division, we are obliged to take the *two* figures 37 these do not mean *thirty-seven*, but $3 \times 9 + 7 =$ *thirty-four:* hence 8 in 37 will go 4 times with 2 over; 8 in 27 (not *twenty-seven*, but $2 \times 9 + 7 =$ *twenty-five*) will go 3 times with 1 over; and so on.

Ex. 79.

1. Express 1828, 347·05 in the septenary scale.
2. Express 300 in the scales of 2, 3, 4, 5, 6.
3. Express 10000 in the scales of 7, 8, 9, 11, 12.
4. Transform 444 and 4321 from the quinary to the septenary.
5. Transform 27t and 7007 from the undenary to the octenary.
6. Transform 123 and 10000 from the nonary to the quaternary.

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189. The common processes of Arithmetic are carried on with these, as with ordinary Nos., observing that when we have to find what Nos. we are to *carry* in Addition, &c., we must not now divide by 10, but by the radix of the scale in question.

	Addition.		*Subtraction.*	
	$r=4$	$r=7$	$r=3$	$r=12$
Ex. 1.	32123	65432	201210	7t348
	21003	54321	102221	5e6t4
	33012	43210	21212	1t864
	22033	1444		
	31102	65001		
	332011	226041		

Ex. 2. Multiply together 68 and 71 in the undenary scale; express also and multiply these Nos. in the nonary scale, and compare the results, by reducing each to the other scale.

Here 68 and 71 in the undenary = 82 and 86 in the nonary:

```
  68       82       9) 4378          11) 7823
  71       86        9) 533...3       11) 642...8
  68      543         9) 65...2        11) 52...7
 431      727             7...8              4...3
4378     7823
```

It will be seen that in the last two operations we have shewn that 4378 in the undenary = 7823 in the nonary, and *vice versâ*, as it should be.

Ex. 3. Divide 234431 by 414 (quinary), and extract the square root of 122112 (senary).

```
414) 234431 (310         1̇22̇11̇2 (252
     2302                 4
      423             45) 421
      414                 401
       41            542) 2012
                          1524
                            44
```

There is a rem^r here in each case.

Ex. 80.

1. Take six terms of the series 1, 10, 10^2, &c.; express and add them in the senary scale, and reduce the result to the denary.

2. Multiply the common Nos. 64 and 33 in the binary and quatenary, and transform each result to the other scale.

3. Transform 1756 and 345 from the octenary scale to the nonary; multiply them in both scales, and divide the result in each case by the first of the two numbers.

4. Divide 51117344 by 675 (octenary), 37542627 by 42*t* (undenary), and 29*t*96580 by 2*tt*9 (duodenary).

5. Extract the square roots of 25400544 (senary), 47610370 (nonary), and 32*e*75721 (duodenary).

6. Express in common Nos. the greatest and least that can be formed with four figures in the scales of 6, 7, and 8.

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190. A decimal fraction may be considered as a vulgar fraction, whose den^r^ is some power of 10, the No. of decimal places pointed off from the right being the same as the index of the den^r^. Hence, if P represent the digits, or, as they are called, the *significant part*, of a decimal of p places, its equivalent vulgar fraction $N = \frac{P}{10^p}$.

It is obvious that decimals, having the same significant part, P, may differ much in value, in consequence of the difference in the value of p, *i. e.* in the position of their decimal points.

$$\text{Thus } 1.23 = \frac{123}{10^2},\ .0123 = \frac{123}{10^4},\ 12.3 = \frac{123}{10}.$$

191. *To prove the rule for pointing in Mult^n^ of Decimals.*

Let M and N be two fractions, which, expressed as decimals, give the significant parts P and Q, with p and q places of decimals respectively; then

$$M = \frac{P}{10^p},\ N = \frac{Q}{10^q},\ \text{and } M \times N = \frac{P}{10_p} \times \frac{Q}{10^q} = \frac{PQ}{10^{p+q}}.$$

Now, $\frac{PQ}{10^{p+q}}$ represents a decimal, whose significant part is PQ (the product of the two decimals as whole Nos.) and having $p+q$ decimal places; hence the rule:

Multiply as in whole Nos.; and in the product point off as many decimal places as there are in the Multiplier and Multiplicand together.

192. *To prove the rule for pointing in Div^n of Decimals.*

Let M, N, P, Q, p, q be the same as before;

$$\text{then } \frac{M}{N} = \frac{P}{10^p} \div \frac{Q}{10^q} = \frac{P}{10^p} \times \frac{10^q}{Q} = \frac{P}{Q} \cdot \frac{10^q}{10^p};$$

$$\text{hence } \frac{M}{N} = \frac{P}{Q} \cdot \frac{1}{10^{p-q}}, \text{ or } = \frac{P}{Q}, \text{ or } = \frac{P}{Q} \cdot 10^{q-p}, \text{ as } p \gtreqless q.$$

Now $\frac{P}{Q}$ is the quotient obtained by dividing P by Q, as in whole Nos.; hence the rule:

Divide as in whole Nos.; then

(i) *If the No. of places in the dividend exceed that in the divisor, point off in the quotient a No. of decimal places equal to that excess;*

(ii) *If the No. in the dividend be the same as that in the divisor, the quotient will have no decimal places;*

(iii) *If the No. in the dividend fall short of that in the divisor, annex to the quotient a No. of cyphers equal to that defect.*

Notice that any cyphers, *annexed to the dividend* in the process of Division, must be reckoned as so many decimal places: thus $1 \div 12.5 = \frac{1.000}{12.5} = .08$.

193. *To prove the rule for reducing a circulating decimal to a vulgar fraction.*

We need here consider only the *fractional* part of a circulating decimal. If there be any figures *before* the decimal point, these may be kept separate, and connected with the vulgar fraction equivalent to the other part, so making a mixed No.

Let N be a circulating decimal, in which P represents the figures *not* recurring, and Q the *period* or recurring part; and let P and Q contain p and q digits respectively.

Then $N = .PQQ$ &c. and $10^p \,.N = \quad P.QQQ$ &c.

and $10^{p+q}.N = PQ.QQQ$ &c.

$$\therefore (10^{p+q} - 10^p)N = PQ - P,$$

$$\text{or } N = \frac{PQ-P}{10^{p+q}-10^p} = \frac{PQ-P}{10^p(10^q-1)}.$$

Hence the rule—(since $10^q - 1$ will be expressed by q *nines*, and 10^p is 1 followed by p cyphers)—

For the numerator, set down the decimal to the end of the first period, and subtract from it the non-recurring part; and for the denominator, set down as many 9'*s as there are recurring figures, followed by as many cyphers as there are non-recurring figures.*

194. Let $\frac{a}{b}$ be a proper fraction in its *lowest terms.*

Then if b can be but in the form $2^m\, 5^n$, *i. e.* the product of any powers of 2 and 5, the fraction may be reduced to a *terminating* decimal, in which the number of places will be the greater of the two, m and n.

For if $m > n$, then $\frac{a}{2^m 5^n} = \frac{a.5^{m-n}}{2^m 5^m} = \frac{a.5^{m-n}}{10^m}$,

which, expressed as a decimal (190), has m decimal places; and if $m < n$, then $\frac{a}{2^m 5^n} = \frac{a.2^{n-m}}{2^n 5^n} = \frac{a.2^{n-m}}{10^n}$,

which, expressed as a decimal, has n decimal places.

195. If b be not of the form $2^m 5^n$, the fraction cannot be reduced to a terminating decimal.

For here no factor, by which we could multiply both numerator and denominator, will make the denominator a power of 10; since all powers of 10 contain only factors 2 and 5, whereas the denominator here contains some factor different from these.

In such a case it may be shewn that the figures of the decimal will recur, and the No. of figures in the period will be less than b.

196. *To find the amount of a given sum, in any given time, at Simple Interest.*

Let P be the principal in *pounds*, n the length of time in *years*, r the interest of £1 for 1 year; then the interest of P pounds for 1 year will be Pr, and for n years, will be Prn, which is the whole interest required; and the *Amount*, $M = P + Prn = P(1 + rn)$.

If $M = 2P$, or the original sum has doubled itself, we have $2P = P(1 + rn)$, and $n = 1 \div r$, $r = 1 \div n$.

Thus at 4 per cent., since here we should have $r = \frac{4}{100}$, and $\therefore n = \frac{100}{4} = 25$, it appears that any given sum will double itself in 25 years; but to have doubled itself in 15 years, it should be put to interest at $6\frac{2}{3}$ per cent., since then we should have $n = 15$, and $\therefore r = \frac{1}{15}$, and $100r = 6\frac{2}{3}$.

Cor. Hence the *Simp. Int.* on any sum, is proportional, (i) to the *Principal*, when the Rate and Time are given, (ii) to the *Rate*, when the Principal and Time are given, (iii) to the *Time*, when the Principal and Rate are given (*Arithmetic*, 96); but the *Amount* only in the first case.

197. *To find the Amount of a given Sum, in any given time, at Compound Interest.*

Let P, n denote, as before, the Principal and Time; R the amount of £1 with its interest for 1 year $= 1 + r$;

then PR will be the amount of £P with interest for 1 year, and this becomes the *Principal* for the 2nd year: $\therefore PR \times R = PR^2$ will be the amount of £P for 2 years, and this becomes the *Principal* for the 3rd year: $\therefore PR^2 \times R = PR^3$ will be the amount of £P for 3 years, &c., hence $M = PR^n = P(1+r)^n$, the amount of £P for n years: and the interest $= PR^n - P = P(R^n - 1)$.

COR. Hence the *Comp. Int.* on any sum, as also the *Amount*, is proportional to the *Principal*, when the Rate and Time are given; but the corresponding statement will not hold good, for the other cases of (196 Cor.).

198. *To find the present Value and Discount on any sum for a given time,* (i) *at Simple* (ii) *at Compound Interest.*

Let V represent the present value, D the discount, of a sum P due at the end of n years; then, since V is the sum, which at Int. for the given time will amount to P, we have (i) $P = V(1+rn)$, (ii) $P = V(1+r)^n$; hence

(i) $V = \dfrac{P}{1+rn}$, and $D = P - V = \dfrac{Prn}{1+rn}$, (ii) $V = \dfrac{P}{(1+r)^n}$.

Ex. 1. What sum will in 9 months amount to £600, at 5 per cent. per annum, Simple Interest?

Here $M = 600$, $r = \frac{5}{100} = .05$, $n = \frac{3}{4} = .75$, to find P:

$$\therefore P = \frac{M}{1+rn} = \frac{600}{1 + .05 \times .75} = \frac{600}{1.0375} = £578\ 6s\ 3d \text{ nearly.}$$

Ex. 2. In what time will £91 13*s* 4*d* amount to £100 at 3 per cent., Simple Interest?

Here $P = 91\frac{2}{3}$, $r = \frac{3}{100}$, $M = 100$, to find n:

$\therefore 100 = 91\frac{2}{3}\ (1 + \frac{3}{100}n)$, whence $n = \frac{100}{33} = 3\frac{1}{33}$ years.

Ex. 3. Find the Comp. Int. on £275 for 3 years at 5 per cent.

Here $P = 275$, $n = 3$, $R = 1.05$, to find M:

$$\therefore M = 275 \times (1.05)^3 = £318\ 6s\ 11\tfrac{1}{4}d,$$

and Interest $= M - P = £43\ 6s\ 11\frac{1}{4}d$.

MISCELLANEOUS EXAMPLES: PART I.

1. Multiply $a^2 - 2ax - b^2 + bx$ by $b^2 + ax$.
2. Divide $3x^3 + 4abx^2 - 6a^2b^2x - 4a^3b^3$ by $2ab + x$.
3. If $x = 1$, $y = -2$, $z = 3$, find the value of
$$\frac{3x^2 - 2xy + 5y^2 + 5z^2 + 2yz + 2xz}{4x^2 + 2xy + 3y^2 + 2z^2 + yz - xz}.$$
4. Reduce $\dfrac{m^3a^2 + n^3a^2}{a(m^2 + n^2) - man}$ and $\dfrac{x^4 + x^2y^2 + y^4}{x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4}$.
5. Extract the square roots of $1\frac{56}{169}$ and 66.455104.
6. Simplify $\dfrac{(1\frac{1}{2} - x) - \frac{1}{2}(x - 1\frac{1}{3})}{(1\frac{1}{3} - x) - \frac{1}{3}(x - 1\frac{1}{2})}$ and $\dfrac{1}{x-1} - \dfrac{1}{2x+2} - \dfrac{x+3}{2x^2+2}$.
7. Sum the A. P. $7 + 8\frac{1}{2}$ &c. to 8 and to n terms.
8. Insert an H. mean between $1\frac{1}{2}$ and $1\frac{1}{3}$.
9. Reduce to their simplest forms $\sqrt{125}$, $\sqrt{98a^2x}$, $\frac{4}{7}\sqrt[3]{21\frac{7}{16}}$.
10. Expand $(1 - 2x)^{-\frac{1}{2}}$ to five terms.
11. (i) $\frac{1}{3}(5x - 7) - \frac{1}{5}(4x - 9) = 3\frac{4}{5}$ (ii) $x + 7 = \sqrt{5x^2 + 19}$

    (iii) $\left.\begin{array}{l}\frac{1}{2}x - \frac{1}{3}y = 1 \\ 6(x + y) - 3(x - y) = 13(x - 1)\end{array}\right\}$ (iv) $\left.\begin{array}{l}x^2 + y^2 = 13 \\ xy = 6\end{array}\right\}$
12. A certain fraction becomes 1 when 3 is added to the num^r and $\frac{1}{2}$ when 2 is added to the den^r: find it.

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13. Write down the square of $1 + 2x - x^2 - \frac{1}{2}x^3$.
14. Divide $51x^2y^2 + 10x^4 - 48x^3y - 15y^4 + 4xy^3$ by $4xy - 5x^2 + 3y^2$.
15. Find the value of $x^4 - 2a(a-b)x^2 + (a^2 + b^2)(a - b)x - a^2b^2$, when $a = 1$, $b = -2$, $x = 3$.
16. Find the G. C. M. of
$3x^4 - x^2y^2 - 2y^4$ and $10x^4 + 15x^3y - 10x^2y^2 - 15xy^3$.
17. Extract the cube roots of 1953125 and 5.
18. Simplify $\dfrac{2(x^2 - \frac{1}{4})}{2x + 1} + \frac{1}{2}$ and $\dfrac{a^3 + 3a^2x + 3ax^2 + x^3}{x^3 - y^3} \div \dfrac{(a + x)^2}{x^2 + xy + y^2}$.
19. Sum the G. P. $3 - 1 +$ &c. to 5 terms and *ad infinitum*.
20. Simplify $\{(a^{\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{3}{4}})^{-\frac{1}{2}}\}^{-6}$ and $x^{-1}y^{-\frac{1}{2}}z\sqrt{(xyz^{-\frac{2}{3}})}$.
21. Expand to five terms $\dfrac{1}{\sqrt[3]{a - 3x}}$.

22. Express 3000 (*quaternary*) in the quinary scale, and 3000 (*quinary*) in the quaternary, and all four in the septenary.

23. (i) $\frac{3x-2}{2x-5} - \frac{21-3x}{5} = \frac{6x+13}{10}$ (ii) $\frac{3}{4}x = 1 + \frac{1}{x}$

(iii) $\left.\begin{array}{r} 1\frac{1}{2} - 5\left(\frac{1}{2}x - 1\right) = 2 - \frac{3}{2}(y+1) \\ \frac{2}{3}x + 8 - \frac{1}{2}(y-5) = 11x - 3\frac{1}{3}(3x-2) \end{array}\right\}$

24. A can do a piece of work in $10\frac{1}{2}$ days which A and B can do together in $5\frac{3}{5}$ days: how long would B take to do it alone.

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25. Find the product of $x^2 - a$, $x^2 - a^{\frac{1}{2}}x + a$, and $x^2 + a^{\frac{1}{2}}x + a$.

26. Divide $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$ by $\frac{1}{2}x^2 - x + 3$.

27. If $x = 1$, $y = -2$, $z = 3$, find the value of

$$\tfrac{1}{2}\left[x - \tfrac{1}{3}\left\{y - \tfrac{1}{4}(z - x - 2y)\right\}\right].$$

28. Reduce $\frac{x^4 + x^2y}{x^4 - y^2}$ and $\frac{27a^5x^2 - 18a^4x^2 - 9a^3x^2}{36a^6x^2 - 18a^5x^2 - 27a^4x^2 + 9a^3x^2}$.

29. Simplify $\frac{1 - \frac{1}{2}\left\{1 - \frac{1}{3}(1-x)\right\}}{1 - \frac{1}{3}\left\{1 - \frac{1}{2}(1-x)\right\}}$ and $\frac{x+2}{2(x+1)} + \frac{2-x}{2(x-1)} - \frac{x}{x^2+1}$.

30. Find the square roots of 19321, 1.9321, and 19.321.

31. Obtain a fourth proportional to $\frac{2}{7}$, $\frac{3}{4}$, $\frac{5}{6}$, and a mean proportional to .017 and .153.

32. Sum the G. P. $\frac{4}{3} - \frac{2}{3} +$ &c. to n terms and *ad infinitum.*

33. Expand $(ax - x^2)^{-\frac{3}{5}}$ to five terms.

34. In how many ways may a sum of 40 guineas be paid in dollars (4*s* 6*d*) and doubloons (13*s*)? and how may it be paid with fewest coins?

35. (i) $\frac{x-2}{3} - \frac{1-\frac{1}{2}x}{6} = 87\frac{1}{4} - \frac{27(x-2)}{5}$

(ii) $\left.\begin{array}{r} \frac{1}{2}x - 12 = \frac{1}{4}y + 8 \\ \frac{1}{5}(x+y) + \frac{1}{3}x = \frac{1}{4}(2y-x) + 35 \end{array}\right\}$ (iii) $\frac{13}{x+2} + \frac{4}{x} = 3\frac{14}{15}$.

36. A can correct 70 pages for the press in $1\frac{1}{2}$ hr, B can correct 150 pages in $2\frac{1}{4}$ hrs: how long will they be in correcting 425 pages jointly?

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37. Multiply $(a + b + c)(a + b - c)$ by $(a - b + c)(b + c - a)$.

38. Divide $1 - \frac{1}{2}x$ by $1 - \frac{1}{3}x - \frac{1}{4}x^2$ to five terms.

39. If $a = -x = \frac{1}{2}$, b = 0, find the numerical value of

$$x^4 - (a - b)x^3 + (a - b)b^2x - b^4.$$

40. Reduce to its lowest terms $\frac{2x^3 - x^2 + x + 1}{2x^3 + 3x^2 + 3x + 1}$.

41. Find the cube roots of 2685619 and $\frac{1}{3}$.

42. Simplify the fraction $\dfrac{\frac{1}{2}(x+1\frac{1}{3})-\frac{2}{3}(1-\frac{3}{4}x)}{1\frac{3}{4}-\frac{1}{3}(x+4\frac{1}{4})}$.

43. Expand $(a^4-4a^2x^2)^{\frac{7}{4}}$ to five terms.

44. Reduce to their simplest forms $\dfrac{2a}{3}\sqrt[3]{\dfrac{9}{4a^2}}$ and $\dfrac{3x}{2}\sqrt[4]{\dfrac{80y^2}{81x^2}}$.

45. Sum the A. P. $\frac{1}{2}+\frac{2}{3}+$ &c. to 31 and to $n-2$ terms.

46. Transform 1828 into the septenary scale, and square it; reduce the result to the nonary, and extract the square root; and express the latter two results in the denary.

47. (i) $3x-\frac{1}{2}(x-1\frac{1}{2})=9-\frac{1}{4}(5x-7)$

(ii) $\left.\begin{array}{r} x-y-z=6 \\ 3y-x-z=12 \\ 7z-y-x=24 \end{array}\right\}$ (iii) $\left.\begin{array}{r} a(x+y)-b(x-y)=2a^2 \\ (a^2-b^2)(x-y)=4a^2b \end{array}\right\}$

48. Two men can do a piece of work in 12 days, and one of them can do half as much again in 24 days: in what time could the other do a third as much again?

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49. Simplify $\frac{1}{4}\{\frac{1}{5}a-(b-a)\}-\frac{1}{2}[(b-\frac{1}{3}a)-\frac{2}{3}\{a-\frac{3}{4}(b-\frac{4}{5}a)\}]$.

50. If $a=1$, $b=3$, $c=5$, find the numerical value of

$$\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2.$$

51. Expand and simplify the quantities in the preceding question.

52. Find the G. C. M. of

$7x^3-2x^2y-63xy^2+18y^3$ and $5x^4-3x^3y-43x^2y^2+27xy^3-18y^4$.

53. Extract the square roots of 1110916 and $9+2\sqrt{14}$.

54. Simplify

$$\left(a+b+\frac{b^2}{a}\right)\div\left(a+b+\frac{a^2}{b}\right) \text{ and } \left(a-b+\frac{b^2}{a+b}\right)\div\left(a+b+\frac{b^2}{a-b}\right)$$

55. Sum .2 + .02 + .002 to n terms and *ad infinitum.*

56. How many terms of the series 17, 15, &c. will make 72?

57. Expand $(a^2-bx)^{-\frac{2}{5}}$ to five terms.

58. How many different throws can be made with two dice?

59. (i) $\dfrac{3}{x+1}=8-2\left(\dfrac{4x+3}{x+3}\right)$ (ii) $\left.\begin{array}{r} 5x+7y=43 \\ 11x+9y=69 \end{array}\right\}$

(iii) $x^2y-xy^2=6=2xy$.

60. A person bought cloth for £12: if he had bought one yard less for the same money, each yard would have cost him 1*s.* more; how many yards did he buy?

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61. Multiply $2y + 3x^{\frac{1}{4}}y^{\frac{1}{2}} - x^{\frac{1}{2}}$ by $7x^{\frac{1}{4}} - 5y^{\frac{1}{2}}$.
62. Divide $x^4 + 4x + 3$ by $x^2 - 2x + 3$.
63. If $a = 1$, $b = 2$, $c = 3$, find the value of $\sqrt[3]{a(b^2 + ac) - \frac{1}{2}b^2c}$.
64. Find the G. C. M. of $a^3(b^4 - b^2c^2)$ and $b^3(ab + ac)^2$.
65. Obtain the fourth root of $16x^3(x - 2) - 8x^2(x^{-1} - 3) + 1$.
66. Simplify $\frac{x + 1}{2x - 1} - \frac{x - 1}{2x + 1} - \frac{1 - 3x}{x(1 - 2x)}$.
67. Find the G. mean between $12\frac{1}{2}$ and 13, to 3 places of decimals.
68. Expand $\left(\frac{1}{a^2x - ax^2}\right)^{\frac{1}{2}}$ to five terms.
69. What number is that, which is just as much below 35 as its half is above its third part?
70. Convert 297 to radix 11: square and cube it in that scale, extract the roots, and reconvert them to the common scale.
71. (i) $\frac{1}{4}(3x + 5) - \frac{1}{3}(21 + x) = 39 - 5x$.
(ii) $2x^2 + x = 28$. (iii) $2x - 9y + 2 = 0 = 3x - 12y + 2\frac{1}{2}$.
72. *A* and *B* can reap a field in 10 hrs, *A* and *C* in 12 hrs, *B* and *C* in 15 hrs: in what time can they do it *jointly* and *separately?*

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73. Obtain the quotient of $6\sqrt[3]{x^4} - 96\sqrt[3]{x^{-4}}$ by $\sqrt[3]{x} - 2\sqrt[3]{x^{-1}}$.
74. If $x = \frac{1}{2}$, and $x + y = x + y + z = 0$, find the value of
$$(y^2 - z^2)\{y^2 + z^2 - y(x - z)\}.$$
75. Reduce $\frac{x(x^3 + y^3)(x - y)}{(x^2 - y^2)(x^2 + y^2 - xy)}$ and $\frac{x^4 + 3x^3 + x + 3}{x^3 - 8x + 3}$.
76. Add together $7\sqrt{63} + 2\sqrt{252} + 11\sqrt{28}$.
77. Find $\sqrt{3.14159}$, and the fourth root of $x^4 - \frac{1}{2}x + \frac{3}{2}x^2 + \frac{1}{16} - 2x^3$.
78. Shew by the Bin. Theor. that $\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \&c.$
79. Sum the A. P. $\frac{4}{5} + 2 + \&c.$ to 9 and to n terms.
80. Form the equation whose roots are $2, -2, 1 + \sqrt{5}, 1 - \sqrt{5}$.
81. What number is that which is the same multiple of 7, that its excess above 20 is of its defect from 30?
82. How many different arrangements can be made of the letters of the word *Novogorod?* How many with two o's at the beginning and two at the end?
83. (i) $\frac{1}{3}(7x + 5) - \frac{4}{5}(x + 4) + 6 = \frac{3}{2}(x + 3)$
(ii) $x + y - 8 = 0 = \frac{1}{2}(x - y) + \frac{2}{3}(x - \frac{1}{2}y + 2)$
(iii) $x + \sqrt{5x + 10} = 8$.

84. Out of £5000, a person leaves £20 to an old servant, and the remainder among three societies, A, B, and C, so that B may have twice as much as C, and A three times as much as B: how much does each receive?

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85. Multiply $\sqrt{x^3} + 1 + \dfrac{1}{\sqrt{x^3}}$ by $\sqrt{x^3} - 1 + \dfrac{1}{\sqrt{x^3}}$.

86. Divide $\frac{1}{2}a^3 + \frac{3}{2}a^2x - 2x^3$ by $\frac{1}{2}a + x$.

87. If $a = 1$, $b = \frac{2}{3}$, $x = 7$, $y = 8$, find the numerical value of
$5(a-b)\sqrt[3]{(a+x)y^2} + a - b\sqrt{(a+x)y} - \sqrt[4]{\tfrac{1}{2}y^2 - \{a - \sqrt{3(x+2b)}\}^2}$.

88. Simplify $1\frac{1}{2} - \frac{3}{4}\{1 - \frac{2}{3}(x - \frac{1}{2})\}$ and $\{a - \frac{1}{2}(a - \frac{2}{3}b)\} \div \{b - \frac{1}{3}(a + \frac{3}{3}b)\}$.

89. Write down the quotient of $ax^{-1} + b^2$ by $a^{\frac{1}{3}}x^{-\frac{1}{3}} + b^{\frac{2}{3}}$.

90. Find the square root of $(x + x^{-1}) - 2(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) + 1$.

91. Sum the A. and G. P. $\frac{4}{5} + 2 +$ &c., each to n terms. Can the latter series be summed *ad infinitum?*

92. Expand $\sqrt[4]{1 + 4x}$ to five terms, and square the result.

93. Find two numbers in the ratio of $1\frac{1}{2} : 2\frac{2}{3}$, such that, when increased by 15, they shall be in the ratio of $1\frac{2}{3} : 2\frac{1}{2}$.

94. In how many ways may £24 16*s* be paid in guineas and crowns?

95. (i) $\frac{1}{2}(9x + 7) - \{x - \frac{1}{7}(x - 2)\} = 36$

(ii) $\left.\begin{aligned} x + 1 : y :: 5 : 3 \\ \tfrac{2}{3}x - \tfrac{1}{2}(5 - y) = 3\tfrac{5}{12} - \tfrac{1}{4}(2x - 1) \end{aligned}\right\}$

(iii) $\dfrac{8 - x}{2} - \dfrac{2x - 11}{x - 3} = \dfrac{x - 2}{6}$.

96. A messenger starts with an errand at the rate of $3\frac{2}{3}$ miles an hour; another is sent half-an-hour after to overtake him, which he does in 2 hours: at what rate did he ride? Find also in what time he will do it, if he rides 12 miles an hour.

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97. Simplify $\frac{1}{6}\{x(x+1)(x+2) + x(x-1)(x-2)\} + \frac{2}{3}(x-1)x(x+1)$.

98. Divide $a^4 - \frac{19}{6}a^2b^2 + \frac{1}{3}ab^3 + \frac{1}{6}b^4$ by $a^2 + 2ab + \frac{1}{3}b^2$.

99. Find the G. C. M. of $3x^3 + 4x^2 - 3x - 4$ and $2x^4 - 7x^2 + 5$.

100. Reduce $\dfrac{(x^4 - b^4)(x - b)}{(x^2 + b^2 - 2bx)(bx + x^2)}$ and $\dfrac{x + \sqrt[4]{a^2x^2y}}{x - a\sqrt{y}}$.

101. Find the cube root of 69.426531.

102. Multiply $1 + a^{\frac{2}{3}} - x^{-\frac{2}{3}} + a^{\frac{4}{3}} + x^{-\frac{4}{3}} + a^{\frac{2}{3}}x^{-\frac{2}{3}}$ by $x^{-\frac{2}{3}} - a^{\frac{2}{3}} + 1$.

103. Find the common difference of an A. P., when the first term is 1, the last term 50, and the sum 204.

104. If $a : b :: c : d$, shew that $7a + b : 3a - 5b :: 7c + d : 3c - 5d$.

105. Divide 100 into two parts, so that $\frac{1}{3}$ the greater may be greater than $\frac{1}{2}$ the less by $\frac{1}{4}$ their difference.

106. Employ the septenary scale to find the side of a square which contains a million square feet.

107. (i) $\frac{1}{6}(x+3) - \frac{1}{7}(11-x) = \frac{2}{5}(x-4) - \frac{1}{21}(x-3)$

(ii) $\frac{2x-1}{2x+1} + \frac{2x+1}{2x-1} = 3.$ (iii) $\left.\begin{array}{l} 3x - y + z = 17 \\ 5(x+y-2) = 2(y+z) \\ 4(x+y+z) = 3(1-x+3z) \end{array}\right\}$

108. A and B engaged in trade, A with £275, B with £300; A lost half as much again as B, and B had then remaining half as much again as A: how much did each lose?

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109. If $a - b = x = 3$ and $a + b + x = 2$, find the value of $(a-b)\{x^3 - 2ax^2 + a^2x - (a+b)b^2\}$.

110. Shew that $(2a + b^{-1})(2b + a^{-1}) = (2a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{-\frac{1}{2}}b^{-\frac{1}{2}})^2$.

111. Find the L. C. M. of $6x^2 - 13x + 6$, $6x^2 + 5x - 6$, and $9x^2 - 4$.

112. Obtain the square root of $\frac{1}{4}x^4 + \frac{1}{9}a^4 - \frac{1}{3}ax(2a^2 + 3x^2 - 4ax)$.

113. Obtain $\sqrt{6}$ to four places, and thence find $\sqrt{\frac{1}{6}}$, $\sqrt{\frac{2}{3}}$, $\sqrt{1\frac{1}{2}}$.

114. Simplify $\frac{a^{\frac{1}{4}(3m-1)}b^{-1}}{a^{\frac{1}{4}(m-3)}b^{\frac{1}{2}(m-1)}}$ and $\frac{b}{b+x} + \left(\frac{x}{b+x}\right)^2 + \frac{bx}{(b+x)^2}$

115. Square $a - 2b - 3c$ and $2a - \frac{1}{2}bx - \frac{1}{4}cx^2 + 2dx^3$.

116. Sum the G. P. $5 + 2 + \&c.$ to n terms and *ad infinitum*.

117. The trinomial $ax^2 + bx + c$ becomes 8, 22, 42 respectively, when x becomes 2, 3, 4: what does it become when $x = -\frac{1}{3}$?

118. Expand $\sqrt{1-4x}$ to five terms, and obtain the same by Evol^n.

119. (i) $\frac{1}{9}(4x-21) + 3\frac{3}{4} + \frac{1}{4}(57-3x) = 241 - \frac{1}{12}(5x-96) - 11x$

(ii) $11x^2 + 1 = 4(2-x)^2$

(iii) $\left.\begin{array}{l} \frac{1}{5}(3x-2y+1) - \frac{1}{3}(x-y) = \frac{4}{9}y \\ \frac{5}{x} - \frac{3}{2y} = \frac{15}{2xy} \end{array}\right\}$

120. A and B sold 130 ells of silk, of which 40 were A's and 90 B's, for 42 crowns; and A sold for a crown $\frac{1}{3}$ an ell more than B did. How many ells did each sell for a crown?

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121. Write down the quotient of $16 - 81a$ by $2 + 3\sqrt[4]{a}$.

122. Multiply $a^2 + \frac{2}{3}(a + b)x - \frac{1}{2}x^2$ and $a^2 - \frac{2}{3}(a - b)x + \frac{1}{2}x^2$.

123. Reduce to its lowest terms $\dfrac{9x^3 + 6x^2 - 2x - 4}{12x^3 - 5x^2 + 4x - 4}$.

124. Find the L. C. M. of $a^2 \pm x^2$, $(a \pm x)^2$, and $a^3 \pm x^3$.

125. Obtain the square root of $1\frac{1}{3}$ and of $12 + 6\sqrt{3}$.

126. Simplify $a - (b - c) - \{b - (a - c)\} - [a - \{2b - (a - c)\}]$, and shew that $\dfrac{a + c}{(a - b)(x - a)} - \dfrac{b + c}{(a - b)(x - b)} = \dfrac{x + c}{(x - a)(x - b)}$.

127. Sum the A. P. $\frac{1}{2} + \frac{1}{3} +$ &c. to 7 and to n terms.

128. How could a sum of £24 16*s* be paid from A to B with the use of fewest coins, if A have only guineas and B crowns?

129. Simplify $\sqrt{8(a^3x + ax^3) - 16a^2x^2}$ and $(\sqrt{a})^{\frac{2}{3} - \frac{1}{6}} - \frac{3}{4}(a^{\frac{5}{2}}b\sqrt{a^{-3}b^{-2}})^{\frac{1}{4}}$.

130. Compare the numbers of combinations of 24 different letters, when taken 7 and 11 together; and also when the letters a, b, c occur in each of such combinations.

131. (i) $\dfrac{6x + 18}{13} - 4\frac{5}{6} - \dfrac{11 - 3x}{36} = 5x - 48 - \dfrac{13 - x}{12} - \dfrac{21 - 2x}{18}$

(ii) $\left.\begin{array}{l} 1 + \frac{5}{6}(y + 5) - \frac{1}{3}(7x - 6) = 10 - \frac{1}{12}(3x - 10 + 7y \\ \frac{1}{9}(12 - x) : 5x - \frac{1}{3}(14 + y) :: 1 : 8 \end{array}\right\}$

(iii) $5x - \dfrac{3(x - 1)}{x - 3} = 2x + \dfrac{3(x - 2)}{2}$.

132. A party at a tavern had a bill of £4 to pay between them, but, two having sneaked off, those who remained had each 2*s* more to pay: how many were there at first?

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133. Shew that $(ac \pm bd)^2 + (ad \pm bc)^2 = (a^2 + b^2)(c^2 + d^2)$, and exemplify this identity when $a = 1 = -d$, $b = 2 = -c$.

134. Obtain the product of $x + 2\sqrt[4]{x^2y} + 2\sqrt{y}$ by $x - 2\sqrt[4]{x^2y} + 2\sqrt{y}$.

135. Divide $x^4 - (a^2 - b - c)x^2 - (b - c)ax + bc$ by $x^2 - ax + c$.

136. Reduce $\dfrac{6a^2 - 13ay + 6y^2}{10a^2 - 9ay - 9y^2}$ and $\dfrac{x^3 + 11x^2 + 30x}{9x^3 + 53x^2 - 9x - 18}$.

137. Find the L. C. M. of $m^3n - mn^3$, $m^2 + mn - 2n^2$, and $m^2 - mn - 2n^2$.

138. Obtain the square root of $a^{\frac{1}{3}} - 2a^{\frac{1}{4}} + 3a^{\frac{1}{6}} - 2a^{\frac{1}{12}} + 1$.

139. If $a : b :: c : d$, express $(b + d)(c + d)$ in terms of a, b, c.

140. Find $\sqrt{24}$, and thence deduce the values of

$$\frac{5}{\sqrt{54}},\ \frac{\sqrt{2}}{\sqrt{3} - \sqrt{2}},\ \frac{2\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}},\ \frac{1 + \sqrt{216}}{7 - \sqrt{6}}.$$

141. Insert two A. and two H. means between 1 and 3.

142. Expand $(1-4x)^{-\frac{1}{4}}$ and $(1-4x)^{-\frac{1}{2}}$ to five terms; and shew that the former series, when squared, coincides with the latter.

143. (i) $\frac{1}{2}x - \frac{1}{3}(x-2) = \frac{1}{4}\{x - \frac{2}{3}(2\frac{1}{2} - x)\} - \frac{1}{3}(x-5)$

(ii) $\dfrac{x}{x-2} + \dfrac{x-9}{x-7} = \dfrac{x+1}{x-1} + \dfrac{x-8}{x-6}$

(iii) $\dfrac{7x+1}{6\frac{1}{2}-3x} = \dfrac{80}{3}\left(\dfrac{x-\frac{1}{2}}{x-\frac{2}{3}}\right)$

144. A farmer bought 5 oxen and 12 sheep for £63, and for £90 could have bought four more oxen than he could have bought sheep for £9: what did he pay for each?

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145. Find the continued product of $(x+a)(x+b)(a-2x)(b-x)$.

146. Write down the square and fourth powers of $a - \frac{1}{2}\sqrt{ax} - 2x$.

147. Simplify $\dfrac{(x^2-4x)(x^2-4)^2}{(x^2-2x)^2}$ and $\dfrac{(a^2-1)(a^6-1)}{(a+1)^2(a^2-a)^2}$.

148. Reduce to its lowest terms $\dfrac{3a^2+2b^2+c^2-5ab-3bc+4ac}{3a^2+4b^2+3c^2-8ab-8bc+10ac}$.

149. Extract $\sqrt[3]{.01}$ to four places of decimals.

150. Obtain the square root of $(x+1)^2 - 4\sqrt{x}(x - \sqrt{x} + 1)$.

151. Determine which is the greater $\sqrt{2} \div \sqrt[3]{3}$ or $\sqrt{3} \div \sqrt[3]{5}$.

152. Sum the G. P. $\frac{1}{2} + \frac{1}{3} +$ &c. to n terms and *ad infinitum*.

153. Given -1 to be a root of the equation $x^4 - 7x^2 - 6x = 0$, find the other three roots.

154. In how many different ways could a farmer lay out a sum of £63, in buying sheep and oxen at 30*s* and £9 respectively?

155. (i) $a(x-b) = b(a-x) - (a+b)x$

(ii) $\dfrac{3}{1-3x} + \dfrac{5}{1-5x} + \dfrac{4}{2x-1} = 0$ (iii) $\left.\begin{array}{l} 2x^2+3xy=26 \\ 3y^2+2xy=39 \end{array}\right\}$

156. *A* and *B* can do a piece of work together in 4 days: *A* works alone for two days, and then they finish it in $2\frac{1}{2}$ days more: in what time could they have done it separately?

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157. Find the value of $\sqrt{\dfrac{1+a}{1-b}} + \sqrt{\dfrac{3(1+2a^2)}{1-b^2}} + \sqrt{a^2-2ab+4b^2}$.

when $a = \frac{1}{4}$, $b = \frac{1}{5}$.

158. Divide $a^2 + 2ab^{\frac{3}{2}} + b^3$ by $a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{2}} + b$.

159. Find the G. C. M. of $x^4+7x^3+7x^2-15x$ and $x^3-2x^2-13x+110$.

160. Simplify $\frac{2}{x-2} - \frac{1}{x+2} - \frac{x+6}{x^2+4}$ and $\frac{\frac{3}{4}(1+\frac{2}{5}x)}{1\frac{1}{2}-\frac{2}{3}(1-\frac{1}{2}x)}$.

161. Multiply together $x-1+\sqrt{2}$, $x+2+\sqrt{3}$, $x-1-\sqrt{2}$, and $x+2-\sqrt{3}$.

162. Find the 7th term of $5+5\frac{2}{3}+6\frac{1}{3}+$&c., and its sum to 16 terms.

163. If $a:b::b:c::c:d$, show that $a:b::\sqrt[3]{a}:\sqrt[3]{d}$; and express $(a+b)(c\times d)$ in terms of b and c.

164. Find the least number which, when divided by 39 and 56, shall leave remainders 16 and 27 respectively.

165. Expand $(1+2x^2)^3$ and $(a+2b)^{\frac{3}{2}}$, each to five terms.

166. Express a million in the senary scale, extract its square and cube roots in that scale, and reduce the results to the denary.

167. (i) $\frac{2}{3}(x-5) - \frac{7}{11}(x-13\frac{1}{3}) = 5 - \frac{1}{5}(7-x)$.

(ii) $\frac{x-ay}{b} = 1 = \frac{ax+y}{c}$ (iii) $\frac{3x+8}{x-4} - \frac{5(12-x)}{2x+3} = 11$.

168. If A's money were increased by half of B's, it would amount to £54; and, if B's present sum were trebled, it would exceed three times the difference of their original sums by £6. What had each at first?

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169. Write down the expression for the product of the square root of the sum of the cubes of the square roots of a and b, by the square of the cube root of the sum of their squares: and find its value approximately, when $a=4$, $b=1$.

170. Multiply $x^{-\frac{3}{2}} + 2x^{-\frac{3}{4}}y^{\frac{1}{2}} + 3y$ by $x^{-\frac{3}{2}} - 2x^{-\frac{3}{4}}y^{\frac{1}{2}} + y$.

171. Simplify $\left\{\left(\frac{a^m}{b^n}\right)^{\frac{p}{m}}\right\}^{-\frac{q}{n}}$, and reduce $\frac{x^5-4x^3+3x}{2x^4-5x^2-3}$.

172. Obtain the square root of $1 - ax^{\frac{1}{2}} - \frac{15}{4}a^2x + 2a^3x^{\frac{3}{2}} + 4a^4x^2$.

173. Find the sum of $\frac{a+b}{x-a} - \frac{b}{x-b}$, and of $1 + \frac{2x+1}{2(x-1)} - \frac{4x+5}{2(x+1)}$.

174. Extract $\sqrt{15}$, and thence obtain the square roots of $\frac{5}{3}$, $\frac{3}{5}$, $2\frac{2}{5}$, $41\frac{2}{3}$.

175. Sum the A. P. $13 + 11\frac{1}{2} +$ &c. to 5 and to n terms, beginning in each case with the *ninth*.

176. If $x = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$, $y = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$, find the value of $x^2 + xy + y^2$.

177. Expand $(1 + \sqrt{x})^{-2}$ to five terms, and obtain from the result the series for $(1 + \sqrt{x})^{-4}$.

178. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, the sum of whose squares is 724.

179. (i) $6x - a : 4x - b :: 3x + b : 2x + a$ (ii) $3(x - \frac{1}{4}) - \frac{x - 1}{x + 2} = 5$

(iii) $(x + 5)^2 + (y + 6)^2 = 2(xy - 24)$, $y = x + 1$.

180. *A* does $\frac{3}{5}$ of a piece of work in 6 days, when *B* comes to help him; they work at it together for $\frac{7}{8}$ of a day, and then *B* by himself just finished it by the end of the day: in what time could they have each done it separately?

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181. Find the continued product of $a + x$, $a + \frac{1}{2}y$, $a - \frac{1}{2}z$; and deduce from the result the value of $a + b)^3$.

182. Multiply $\frac{5}{2}x + 3a^{-\frac{1}{3}}x^{\frac{1}{2}} - \frac{7}{3}a^{-\frac{2}{3}}$ by $2x - a^{-\frac{1}{3}}x^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{2}{3}}$.

183. Simplify $x - \frac{1}{2}\{(1\frac{1}{3} - x) - \frac{1}{3}(2\frac{1}{2} - x) - \frac{2}{3}(1\frac{1}{2} - \overline{x - 2\frac{1}{4}})\}$.

184. Reduce to its lowest terms $\frac{x^4 - x^3 + 3x^2 - 2x + 2}{x^4 - 5x^2 + 6x - 5}$.

185. Find the L. C. M. of $ax^3 - a^{\frac{1}{3}}x$, $ax^3 - 1$, and $ax^3 + 1$.

186. Extract the square root of $a^2b^{-2} + \frac{1}{4}a^{-2}b^2 - a^{-1}b + 2ab^{-1}$.

187. Find the sum of $\frac{1}{2(x - 1)^2} + \frac{1}{2(x - 1)} - \frac{x}{2(x^2 + 1)}$.

188. Multiply together $3\sqrt{8}$, $2\sqrt{6}$, $\sqrt{15}$, $\sqrt{20}$; and find $\sqrt{2 + \frac{1}{2}\sqrt{7}}$.

189. Sum the G. P. $6 - 2 +$ &c. to 7 and to n terms.

190. A watch which is 10′ too fast at noon on Monday gains 3′ 10″ daily: what will be the time by it at 7h 12′ A.M. of the following Saturday?

191. (i) $\frac{1}{14}(3x + \frac{2}{3}) - \frac{1}{7}(4x - 6\frac{2}{3}) = \frac{1}{2}(5x - 6)$.

(ii) $5x + 4y = 38\frac{1}{4} + \frac{1}{4}(3x - y)$, $x = 5\frac{5}{24} - \frac{1}{4}\{\frac{1}{2}(x + y) - \frac{1}{3}(x - y)\}$.

(iii) $\frac{2}{x - 4} - \frac{3}{x - 6} = \frac{5}{x - 2}$.

192. A man and his wife would empty a cask of beer in 16 days; after drinking together 6 days, the woman alone drank for 9 days more, and then there were 4 gallons remaining, and she had drank altogether $3\frac{3}{4}$ gallons. Find the number of gallons in the cask at first.

193. If $4a = 5b = 1$, find the value of

$$\{\tfrac{1}{3}a^{\frac{3}{3}} + a^{-\frac{1}{2}}b^{-1})\}^{\frac{2}{8}} - \sqrt{[\tfrac{1}{3}\{1 - a^{-\frac{3}{2}} - (1 + ab^{-1})^{-\frac{1}{2}}\}]}.$$

194. Find the sum of $\dfrac{x^{3n}}{x^n - 1} - \dfrac{x^{2n}}{x^n + 1} - \dfrac{1}{x^n - 1}\,\dfrac{1}{x^n + 1}$.

195. Simplify the surd expressions

$$\frac{3\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} - 2\sqrt{3}}, \quad \frac{3\sqrt{2} - \sqrt{3}}{\sqrt{3} - \sqrt{\frac{1}{2}}}, \quad \frac{3\sqrt{\frac{1}{3}} + 2\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt{\frac{1}{3}} - \frac{1}{3}\sqrt{\frac{1}{2}}}.$$

196. Reduce $\dfrac{(m^2 - 4a^2)(m^2 + am - 2a^2)}{(m^2 - a^2)(m^2 - am - 2a^2)}$ and $\dfrac{a^3 - a^2x - ax^2 - 2x^3}{a^5 - 2a^4x - ax^4 + 2x^5}$.

197. Find the L. C. M. of $3x^2 - 2x - 1$ and $4x^3 - 2x^2 - 3x + 1$.

198. Sum the series $3 - 2 + 1\frac{1}{3} -$ &c. to n terms and *ad infinitum*.

199. Prove that the sum of any number, n, of consecutive odd numbers, beginning with unity, is a square number.

200. Given $y^2 \propto a^2 - x^2$, and when $x = \sqrt{a^2 - b^2}$, $ay = b^2$, find the value of x when $y = \frac{4}{5}$ b.

201. A person distributed £2 1*s* 8*d* among some poor people, giving $9\frac{1}{2}d$ to each man and $6\frac{1}{2}d$ to each woman: how many men were there, it being known that the whole number was a multiple of 10?

202. Expand $(1 + \sqrt[3]{x})^{-6}$ to five terms, and obtain from the result by Evolution the series for $(1 + \sqrt[3]{x})^{-3}$.

203. (i) $\frac{1}{4}\{1 + \frac{3}{2}(x + 2)\} - \frac{2}{7}\{1\frac{1}{3} - (1\frac{1}{2} - x\} = 1\frac{3}{28}$

(ii) $abx^2 - (a + b)x + 1 = 0$ (iii) $\frac{1}{2}(x+y) = x - y = \sqrt{x + 2y - 1}$.

204. A and B lay out equal sums in trade; A gains £100, and B loses so much that his money is now only $\frac{2}{3}$ of A's; but if each gave the other $\frac{1}{3}$ of his present sum, B's loss would be diminished by one half. What had each at first, aud what would A's gain be now?

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205. Shew that $\frac{1}{4}(x^2 + y^2) + z^2 - \frac{1}{2}xy + xz - yz$ and $(y - z)^2$ become identical when $-x = y = a$.

206. Divide $mpx^3 + (mq - np)x^2 - (mr + nq)x + nr$ by $mx - n$.

207. Multiply $a^{\frac{2}{3}} + a^{-\frac{2}{3}} + 2 - a^{\frac{1}{3}} + a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}} - a^{-\frac{1}{3}} + 1$.

208. Reduce to its lowest terms $\dfrac{x^3 + a^2x^2 - ax - a^3}{x^2 - ax + a^{\frac{1}{2}}x - a^{\frac{3}{2}}}$.

209. Obtain the sum of $\dfrac{1 - 2x}{3(x^2 - x + 1)} + \dfrac{1 + x}{2(x^2 + 1)} + \dfrac{1}{6(x + 1)}$.

210. Find the square root of 49.14290404 and the cube root of 8242408.

211. The 3rd and 13th terms of an A. P. are 3 and $\frac{1}{3}$: find the 14th term, and the sum of 20 terms.

212. Simplify the surd expression $\{ab^{-2} . \sqrt{ab^3} . \sqrt[3]{ab^4} . \sqrt[4]{ab^5}\}^{\frac{1}{3}}$.

213. The forewheel of a carriage makes 6 revolutions more than the hind wheel in 120 yards, and the circumference of one is a yard less than that of the other: find that of each.

214. Transform 1000000 from the quinary to the septenary scale; and extract its square and cube roots in the latter.

215. (i) $\frac{1}{2}(x-1)(x-2) = (x-2\frac{2}{3})(x-1\frac{3}{4})$
(ii) $2x + 3y = 5 = -(2y + 3x)$ (iii) $x^2 + xy = a^2$, $y^2 + xy = b^2$.

216. Find the time in which A and B can do together a piece of work, which they can do separately in m and n days. How long must A work to do what B can in m days?

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217. Find the difference between $(n+2)(n+3)(n+4)$ and $24\{n-\frac{1}{2}(n-1)\}\{n-\frac{2}{3}(n-2)\}\{n-\frac{3}{4}(n-1\frac{1}{3})\}$.

218. Divide $a + b^2 + c^3 - 3\sqrt[3]{ab^2c^3}$ by $a^{\frac{1}{3}} + b^{\frac{2}{3}} + c$.

219. Find the sum of $\dfrac{x-a}{x-b} + \dfrac{x-b}{x-a} - \dfrac{(a-b)^2}{(x-a)(x-b)}$.

220. Find the L. C. M. of $x^3 + x^2y + xy^2 + y^3$ and $x^3 - x^2y + xy^2 - y^3$.

221. Obtain $\sqrt{10}$, and thence derive the values of $\frac{1}{5}\sqrt{\frac{2}{5}}$, $\sqrt{4\frac{4}{9}}$, $\sqrt{2\frac{1}{2}}$, $(\sqrt{5}+\sqrt{2}) \div (\sqrt{5}-\sqrt{2})$, and $(\sqrt{5}-\sqrt{2}) \div (5\sqrt{2}-2\sqrt{5})$.

222. Sum $(1\frac{1}{2})^{-1} + 2^{-1} + (2\frac{2}{3})^{-1} +$ &c. to n terms and *ad infinitum*.

223. Expand $(a^2 + 2x^2)^{-\frac{3}{2}}$ and $(2a - 3x)^{-2}$ each to five terms.

224. A servant agrees with a master for 12 months, on the condition of receiving a farthing the first month, a penny the second, fourpence the third, and so on: what would his wages amount to in the course of the year?

225. Given two roots of the equation $x^5 + 4x = 5x^3$ to be 1 and -2, find the other three roots.

226. A person changed a sovereign for 25 pieces of foreign coin, some of them going 30 to the £, the others 15: how many did he get of each?

227. (i) $2ax^2 + (a-2)x - 1 = 0$ (ii) $ax + 1 = by + 1 = ay + bx$
(iii) $\dfrac{x}{x-1} + \dfrac{x+2}{x+1} = \dfrac{8x-13}{4(x-2)}$.

228. Find the time in which A, B, and C can together do a piece of work, which A can do in m days, B in n days, and C in $\frac{1}{2}(m+n)$ days.

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229. Divide $5y^4 + \frac{7}{2}ay^3 - \frac{107}{12}a^2y^2 + \frac{5}{6}a^3y + \frac{7}{6}a^4$ by $\frac{5}{2}y^2 + 3ay - \frac{7}{3}a^2$.

230. Obtain the products of $\sqrt{x^3} + a\sqrt[4]{x^3+a^2}$ (i) by $\sqrt{x^3} - a\sqrt[4]{x^3+a^2}$, (ii) by $\sqrt{x^3} + a\sqrt[4]{x^3-a^2}$, (iii) by $\sqrt{x^3} - a\sqrt[4]{x^3-a^2}$.

231. Find the G. C. M. of

$$3a^4 - a^2b^2 - 2b^4 \text{ and } 10a^4 + 15a^3b - 10a^2b^2 - 15ab^3.$$

232. Find the L.C.M. of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, and $x^4 - 2x^3 + 2x^2 - 2x + 1$.

233. Simplify $\dfrac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\sqrt{2} + 3\sqrt{\frac{1}{2}}}$ and $\dfrac{\sqrt[3]{x} + \sqrt{a}}{\sqrt[3]{x} - \sqrt{a}} - \dfrac{2a^{\frac{1}{2}}x^{\frac{1}{3}}}{x^{\frac{2}{3}} - a}$.

234. Extract the fourth root of

$$\tfrac{1}{16}x^{\frac{20}{3}} - \tfrac{5}{2}x^5y^{\frac{4}{5}} + \tfrac{75}{2}x^{\frac{10}{3}}y^{\frac{8}{5}} - 250x^{\frac{5}{3}}y^{\frac{12}{5}} + 625y^{\frac{16}{5}}.$$

235. Sum $16\frac{1}{3} + 14\frac{2}{3} + 13 +$ &c. to 11 terms, and $\frac{5}{6}+\frac{5}{9}+\frac{12}{27}+$ &c. to n terms and *ad inf.*; and insert 3 H. means between 1 and 2.

236. Given $y^2 - b^2 \propto x + a$, and when $x = b$, $y = a$, find the value of y when $x = 3a$.

237. Four places lie in the order of the letters A, B, C, D. A is distant from D 34 miles, and the distance from A to B is $\frac{2}{3}$ of that from C to D; also $\frac{1}{4}$ of the distance from A to B is less than thrice the distance from B to C by $\frac{1}{2}$ of the distance from C to D. Find the respective distances.

238. If $(1+x)^n = 1 + A_1x +$ &c., and $(1+x)^n = 1 + B_1x +$ &c., shew, by finding the actual values of A_1, B_1, &c., that

$$A_3 + A_2B_1 + A_1B_2 + B_3 = 0.$$

239. (i) $3x + 20 = 7 - \frac{1}{2}\{3 - \frac{4}{3}(x - 1)\}$ (ii) $\dfrac{m}{x} + \dfrac{n}{y} = a$, $\dfrac{n}{x} + \dfrac{m}{y} = b$

(iii) $\dfrac{6y - 4x}{3z - 7} = \dfrac{5z - x}{2y - 3z} = \dfrac{y - 2z}{3y - 2x} = 1.$

240. If in (228) A work for $\frac{1}{4}(3m-2n)$ days and B for $\frac{1}{4}(3n-2m)$ days, in what time will C finish the work?

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241. Write down the quotient of $x^3 - y^{-3}$ by $x^{\frac{1}{2}} + y^{-\frac{3}{4}}$, and divide $x^3 - 2ax^2 + (a^2 + ab - b^2)x - a^2b + ab^2$ by $x - a + b$.

242. If $a=16$, $b=10$, $x=5$, $y=1$, find the value of $(x-b)(\sqrt{a}-b) + \sqrt{(a-b)(x+y)}$ and $(a-x)^2 - (b-x^2) - \sqrt{(a-x)(b+y)}$.

243. Find the G. C. M. of $300x^3+265x^2+50x+24$ and $60x^2+53x+4$.

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244. Simplify $\dfrac{a^{-1} + a^{-1}b^{-3}}{b - 1 + b^{-1}}$ and $\left\{\dfrac{1}{3} + \dfrac{2x}{3(1-x)}\right\} \times \left\{\dfrac{3}{4} - \dfrac{3x}{2(1+x)}\right\}$.

245. Find $\sqrt{\cdot 6}$, and obtain by means of it the values of $\sqrt{2\frac{2}{3}}$, $\sqrt{4\frac{1}{6}}$, $(\sqrt{3} - \sqrt{2})^2$, and $(2\sqrt{3} + 3\sqrt{2}) \div (3\sqrt{3} - 2\sqrt{2})$.

246. Shew that $\sqrt{\{a^2 + \sqrt[3]{a^4 b^2}\}} + \sqrt{\{b^2 + \sqrt[3]{a^2 b^4}\}} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

247. Divide 48 into nine parts so that each may just exceed that which precedes it by $\frac{1}{2}$.

248. Given the coefficients of the 4th and 6th terms of $(1 + x)^{n+1}$ equal to one another: find n.

249. In the permutations of the first eight letters of the alphabet how many begin with *ab*?

250. Express 12345654321 in the scale of 12, and extract its square root in that scale.

251. (i) $\frac{2}{3}(x - 5) - \frac{3}{11}(x - 13\frac{1}{3}) = 15 - \frac{3}{5}(19 - \frac{1}{3}x)$

(ii) $\left.\begin{aligned} ax - by &= a^2 \\ bx - ay &= b^2 \end{aligned}\right\}$ (iii) $\left(\dfrac{8x - 3}{4x - 1}\right)^2 = \dfrac{4x - 5}{x - 1}$.

252. Find the time in which A, B, C can together do a piece of work, which (i) A can do in m days, and B and C together in $\frac{1}{2}(m + n)$ days, or (ii) A can do in m days, A and B in n, and A and C in $\frac{1}{2}(m + n)$ days.

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253. Find the coefficient of x in $(x+2)(x-6)(x+10)(x-5)$, and of x^4 in $(1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \&c.) \times (1 - \frac{1}{3}x + \frac{1}{5}x^2 - \frac{1}{7}x^3 + \&c.)$.

254. Divide $x^2 + y$ by $x^{-\frac{2}{3}} + y^{\frac{1}{3}}$ and $x^{\frac{3}{4}} - ma^{\frac{1}{2}}x^{\frac{1}{2}} + max^{\frac{1}{4}} - a^{\frac{3}{2}}$ by $x^{\frac{1}{4}} - a^{\frac{1}{2}}$.

255. Find the L.C.M. of $6x^3 - 11x^2 + 5x - 3$ and $9x^3 - 9x^2 + 5x - 2$.

256. Simplify $\dfrac{\frac{2}{3}(1 - \frac{1}{2}x) - x}{1 - \frac{1}{2}(1 + 2x)}$, and reduce $\dfrac{2a^2 + ab - b^2}{a^3 + a^2 b - a - b}$.

257. Find the sum of

$$\frac{a}{b} - \frac{ac}{b(b+c)} - \frac{a}{b} \cdot \frac{b - 2c}{b - c}, \text{ and } \frac{2 - 2x^2}{\sqrt{(1 - x^2)^5}} - \frac{1}{\sqrt{1 - x^2}}.$$

258. A walks at the rate of 3 miles an hour, B starts 2 hours after him at 4 miles an hour: how many miles will A have walked before B overtakes him? Find also how long B should start after A, in order that A, when overtaken, may have walked six miles.

259. Simplify $b\sqrt[3]{8a^6 b} + 4a\sqrt[3]{a^3 b^4} - \sqrt{125a^6 b^4}$.

260 If the first term of an A. P. be 6, and the sum of 7 terms 105, find the common difference, and shew that the sum of n terms : sum of $n-3$ terms :: $n+3 : n-3$.

261. Which is the greater of the ratios
$a+2x : a+3x$ and $a^2+2ax+2x^2 : a^2+3ax+3x^2$?

262. Of 12 white and 6 black balls how many different collections can be made, each composed of 4 white and 2 black balls?

263. (i) $(x-1\frac{2}{3})(x-2\frac{1}{2}) = \frac{1}{2}(1+\frac{1}{3}x)(x-1)$
(ii) $\frac{1}{3}x - \frac{1}{3}y + z = 7$, $\frac{1}{2}x + y - \frac{1}{4}z = 1$, $\frac{1}{3}y + \frac{1}{4}z - x + 10 = 0$.

264. A market-woman bought eggs at two a penny, and as many more at three a penny; and, thinking to make her money again, she sold them at five for twopence. She lost, however, $4d$ by the business: how much did she lay out?

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265. Shew that $(x+x^{-1})^2 - (y+y^{-1})^2 = (xy - x^{-1}y^{-1})(xy^{-1} - x^{-1}y)$, and exemplify this result numerically when $x = \frac{1}{2}$, $y = -\frac{2}{3}$.

266. Find the G. C. M. of $4a^2x^4 + 9a^{\frac{3}{2}}x^3 + 2ax^2 - 2a^{\frac{1}{2}}x - 4$ and $3a^{\frac{3}{2}}x^3 + 5ax^2 - a^{\frac{1}{2}}x + 2$.

267. Find by Evolution $\sqrt{a+6x}$ to five terms, and square the result.

268. Simplify $3a - [b + \{2a - (b-x)\}] + \frac{1}{2} - \dfrac{\frac{1}{2} - 2x^2}{2x+1}$.

269. Find the sum of $\dfrac{1}{2x+2} - \dfrac{4}{x+2} + \dfrac{9}{2(x+3)} - \dfrac{x-1}{(x+2)(x+3)}$.

270. A gamester loses $\frac{1}{3}$ of his money, and then wins $10s$; he loses $\frac{1}{3}$ of this, and then wins £1, when he leaves off as he began. What had he at first?

271. The sum of n terms of the series $21 + 19 + 17 +$ &c. is 120; find the n^{th} term and n.

272. Divide 100 into two parts so that one shall be a multiple of 7 and the other of 11.

273. Into how many different triangles may a polygon of n sides be divided, by joining its angular points?

274. Convert 85 and 257 to the quaternary scale; multiply them in that scale, and reduce the result back to the denary.

275. (i) $\frac{1}{2}x + \frac{1}{3}x - 1 = \frac{1}{4}\{3x - \frac{1}{3}(x-1)\}$
(ii) $ax + y = x + by = \frac{1}{2}(x+y) + 1$ (iii) $3x^2y = 144 = 4xy^2$.

276. A and B can reap a field of wheat in m days, B and C in n days, and A can do p times as much as C in the same time: in what time would the three reap it together?

277. Find the value of $ax + by - c$ when

$$x = \frac{mc - nb}{ma - lb} \text{ and } y = \frac{lc - na}{lb - ma}.$$

278. When $a = 4$, $x = -8$, $y = 1$, shew that

$$a^{-\frac{3}{2}}x^2 + y^{\frac{9}{4}} = (a^{-\frac{1}{2}}x^{\frac{2}{3}} + y^{\frac{3}{4}})\ (a^{-1}x^{\frac{4}{3}} - a^{\frac{1}{2}}\ x^{\frac{2}{3}}\ y^{\frac{3}{4}} + y^{\frac{3}{2}}.$$

279. Reduce to its simplest form

$$\frac{3a^{2}x^2 + 5a^{-1}x - 12}{a^{\ 3}x^3 - 8a^{\ 2}x^2 - 12a^{-1}x + 63}$$

280. Find the L. C. M. of

$$ax^2 - 1,\ ax^2 + 1,\ (a^{\frac{1}{2}}x - 1)^2,\ (a^{\frac{1}{2}}x + 1)^2,\ a^{\frac{3}{2}}x^3 - 1,\ a^{\frac{3}{2}}x^3 + 1.$$

281. Obtain the square root of $x^{\frac{4}{3}} - 4x + 8x^{\frac{1}{3}} + 4$.

282. Simplify $\sqrt[3]{40} - \frac{1}{2}\sqrt[3]{320} + \sqrt[3]{135}$, and $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}}$.

283. Shew that the sum of the cubes of any three consecutive numbers is divisible by three times the middle number.

284. If $a : b :: c : d$, shew that $2a^2 - 3b^2 : 2c^2 - 3d^2 :: a^2 + b^2 : c^2 + d^2$.

285. Two thirds of a certain number of poor persons received 1*s* 6*d* each, and the rest 2*s* 6*d* each: the whole sum spent being £2 15*s*, how many poor persons were there?

286. The No. of Combns of n letters taken 5 and 5 together, in all of which a, b, and c occur, is 21: find the No. of Combns of them taken 6 and 6 together, in all of which a, b, c, d, occur.

287. (i) $\sqrt{\frac{2}{3}x + (1 - x)^2} = \frac{1}{3} - x$. (ii) $\frac{1}{x + 3} + \frac{2}{x + 6} = \frac{3}{x + 9}$.

(iii) $x^2 + xy + y^2 = 37$, $x + y = 7$.

288. A certain number of sovereigns, shillings, and sixpences amount together to £8 6*s* 6*d*, and the amount of the shillings is a guinea less than that of the sovereigns, and $1\frac{1}{2}$ guinea more than that of the sixpences: how many were there of each?

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289. What is the difference of $a\,(b+c)^2 + b\,(a+c)^2 + c\,(a+b)^2$ and $(a+b)\,(a-c)\,(b-c) + (a-b)\,(a-c)\,(b+c) - (a-b)\,(a+c)\,(b-c)$?

290. Prove the preceding result when $a = -\frac{1}{2}$, $b = \frac{1}{3}$, $c = -\frac{1}{4}$.

291. Multiply $1 + \frac{1}{2}a^{-\frac{1}{2}}x + \frac{1}{4}a^{-1}x^2$ by $1 - \frac{1}{2}a^{-\frac{1}{2}}x + \frac{1}{8}a^{-\frac{3}{2}}x^3 - \frac{1}{16}a^{-2}x^4$.

292. Obtain the coefficient of x^6 in $(1 - 2x + 3x^2 - 4x^3 + \&c.)^2$

293. Extract the square roots of $7\frac{9}{16}$, .064, and $31 - 10\sqrt{6}$.

294. Simplify

$$\{(a-b)^2 + 4ab\}^{\frac{1}{2}} \times \{(a+b)^2 - 4ab\}^{\frac{3}{2}} \times \left\{\frac{a^4-b^4}{a-b} + 2ab(a+b)\right\}^{\frac{2}{3}}.$$

295. Given two numbers such that the difference of their squares is double of their snm, shew that their product will be less than the square of the greater by the donble of it.

296. Sum to n terms $\frac{a}{n^2} + \frac{2a}{n^2} + \frac{3a}{n^2} +$ &c. and $\frac{a}{n} + 1 + \frac{n}{a} +$ &c.

297. Required two numbers whose sum shall be triple of their difference, and less than 50 by the greater of the two.

298. The No. of Comb^{ns} of $n+1$ things, taken $n-1$ together, is 36: find the number of Permutations of n things.

299. (i) $(a+x)(b+x) - a(b+c) = a^2cb^1 + x^2$

(ii) $\sqrt{x} + \sqrt{a-x} = 2\{\sqrt{x} - \sqrt{a-x}\}$

(iii) $2x^2 + 3y^2 = 5 = -5(2x+3y)$.

300. A can do a piece of work in two hours which B can do in 4 hours, and B and C together in $1\frac{1}{2}$ hour: in what time could they do it, working all three together?

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301. Divide

$$12x - 20x^{\frac{3}{4}}y^{-\frac{1}{3}} + 27x^{\frac{1}{2}}y^{-\frac{2}{3}} - 18x^{\frac{1}{4}}y^{\cdot} + 4y^{-\frac{4}{3}} \text{ by } 4x^{\frac{1}{2}} - 4x^{\frac{1}{4}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}.$$

302. Find the value of $\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b}$, when $x = \frac{4ab}{a+b}$.

303. Extract the square roots of

18945044881 and $(x+x^{-1})^2 - 4(x-x^{-1})$.

304. Find the G. C. M. of $(b-c)x^2 + 2(ab-ac)x + a^2b - a^2c$ and $(ab-ac+b^2-bc)x + (a^2c + ab^2 - a^2b - abc)$.

305. Simplify

$\sqrt{128} - 2\sqrt{50} + 72 - \sqrt{18}$, and $(5\sqrt{5} - 7\sqrt{2}) \div (\sqrt{5} - 2\sqrt{2})^2$.

306. Find the sum of $\frac{x+y}{2x-2y} - \frac{y-x}{2x+2y} - \frac{x^2-y^2}{x^2+y^2}$.

307. When are the hour and minute hands of a watch first together after 12 o'clock?

308. Expand $(3a^{-\frac{2}{3}} - 2a^{-1}x^{\frac{1}{3}})^{-3}$ to five terms.

309. Sum $\frac{1}{12} + \frac{1}{8} + \frac{1}{6} +$ &c. to 8 and to $3n$ terms; and insert four H. means between $\frac{2}{3}$ and $\frac{3}{2}$.

310. The No. of Comb^{ns} of 10 letters, $r-1$ together : No. of Comb^{ns} of them, $r+1$ together :: 21 : 10: find r.

311. (i) $.03x^2 - 2.7x = 30$

(ii) $\left.\begin{array}{l}(x + a)(y - b) + c = (x - a)(y + b) - c \\ (x + b)(y - a) = (x + a)(y - b)\end{array}\right\}$

(iii) $x + y = ax + by = ax^2 - by^2$.

312. Supposing in (300) A to begin by himself, how long after must B and C begin to help him, so that, when the work is finished, A may have done upon the whole twice as much as C?

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313. Obtain the product of $\sqrt{a} + \sqrt[4]{ax} + \sqrt{x}$,

(i) by $\sqrt{a} - \sqrt[4]{ax} + \sqrt{x}$, (ii) by $\sqrt{x} - \sqrt[4]{ax} - \sqrt{a}$.

314. Find the value of $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$ when x=(i) $\frac{24}{25}a$, (ii) $\dfrac{2ab}{1+b^2}$.

315. Write down the quotient of $16a^3x^2 - y$ by $2a^{\frac{3}{4}}x^{\frac{1}{2}} + y^{\frac{1}{4}}$.

316. Extract the square root of $\frac{9}{4} - \sqrt{5}$, and of

$$25\tfrac{3}{7} - \tfrac{20}{7}xy^{-1} + \tfrac{9}{16}x^{-2}y^2 - \tfrac{15}{2}x^{-1}y + \tfrac{4}{49}x^3y^{-2}.$$

317. Expand $\sqrt{\dfrac{a^2}{a-x}}$ and $\sqrt[3]{\dfrac{ax}{a+x}}$, each to five terms.

318. Multiply together

$1 + 2\sqrt{2}$, $4 - \sqrt{3}$, $\sqrt{2} + \sqrt{3}$, $4 + \sqrt{3}$, $2\sqrt{2} - 1$, $\sqrt{3} - \sqrt{2}$.

319. Find the n^{th} term and the sum of n terms of the A. P.

$$\frac{a-n}{n} + \frac{a-2n}{n} + \frac{a-3n}{n} + \text{\&c.}$$

320. If the sum or difference of two numbers be 1, shew that the difference of their squares is the difference or sum of the numbers respectively.

321. A servant agreed to live with his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2 13*s* 4*d* and his livery: what was it worth?

322. How many different sums might be made of a sovereign, half-sovereign, crown, half-crown, shilling, and sixpence, and what would be the value of them all?

323. (i) $\dfrac{2x+a}{b} = \dfrac{x-b}{a} = \dfrac{3ax + (a-b)^2}{ab}$

(ii) $\dfrac{x + 1\frac{1}{3}}{x+2} - \dfrac{x+12}{2(x+19)} = \dfrac{1}{2}$ (iii) $ax - cy = 0 = ay + bx - cxy$.

324. Two girls carried between them 25 eggs to market: they sold at different prices, but each received the same amount upon the whole: the first would have sold them *all* for 1*s*, the second for 13*d*: how many did they each sell?

325. Write down the square of $1 - \frac{1}{2}x + \frac{1}{3}x^2$, and square the result.

326. Divide $-2x^5y^{-6}+17x^6y^{-4}-5x^7-24x^8y^4$ by $-x^2y^{-5}+7x^3y^{-1}+8x^4y^3$.

327. Find $\sqrt{7}$, and thence $\sqrt{\frac{1}{7}}$, $\sqrt{1\frac{3}{4}}$, $\sqrt{3\frac{1}{2}} \div \sqrt{4\frac{1}{2}}$, $2 \div (4 - \sqrt{7})$.

328. Find the value of $\dfrac{a}{2na - 2nx} + \dfrac{b}{2nb - 2nx}$, when $x = \frac{1}{2}(a + b)$.

329. Simplify $\sqrt[5]{32a^6 - 96a^5x}$, and $\dfrac{3\sqrt{8} - 2\sqrt{12} + \sqrt{20}}{3\sqrt{18} - 2\sqrt{27} + \sqrt{45}}$.

330. Find the sum of $\dfrac{1}{x^3} + \dfrac{1}{x^2} - \dfrac{1}{x} - \dfrac{1}{(x^2+1)^2} + \dfrac{x + 1}{x^2 + 1} - \dfrac{3}{x^2(x^2 + 1)^2}$.

331. If $u=p+q+r$, where p is constant, $q \propto xy$, and $r \propto xy^{-1}$, and when $x = y = 1$, $u = 0$, when $x = y = 2$, $u = 6$, and when $x = 0$, $u = 1$, find u in terms of x and y.

332. Shew by the Bin. Theorem that $\sqrt[3]{3} = 1 + \frac{2}{3} - \frac{4}{9} + \frac{40}{81} - \frac{160}{243} + \&c.$

333. In how many ways could I distribute exactly 55*s* among the poor of a parish, by giving 1*s* 6*d* to some and 2*s* 6*d* to others?

334. How many words can be formed of 4 consonants and 2 vowels, in a language of 24 letters, of which 5 are vowels?

335. (i) $\dfrac{c}{a - c}\left(x + \dfrac{1}{x}\right) = 1 + \dfrac{a + c}{(a - c)x} + \dfrac{b}{a - c}\left(1 + \dfrac{1}{x}\right)$

(ii) $4x - 5y + mz = 7x - 11y + nz = x + y + pz = 3$.

336. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100′: supposing the stream to have a current of 2 miles an hour, find at what rate they would row in still water?

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337. If $x = \dfrac{2ac}{b(1 + c^2)}$, find the value of $\dfrac{\sqrt{a + bx} + \sqrt{a - bx}}{\sqrt{a + bx} - \sqrt{a - bx}}$.

338. Reduce to its lowest terms $\dfrac{3ax^3 - 2a^{\frac{2}{3}}x^2 - a^{\frac{1}{3}}x}{6a^{\frac{2}{3}}x^2 - a^{\frac{1}{3}}x - 1}$

339. Find the coefficient of x^6 in $(1 + \frac{1}{2}x + \frac{2}{3}x^2 + \frac{3}{4}x^3 + \&c.)^2$

340. Find the sum of $\dfrac{a^3 + a^2b}{a^2b - b^3} - \dfrac{a(a - b)}{(a + b)b} - \dfrac{2ab}{a^2 - b^2}$.

341. Simplify $a^3bc\sqrt[5]{a^{-9}bc} - b^2c\sqrt[5]{a^6b^{-4}c} + a^2b^4c^2\sqrt[5]{243a^{-4}b^{14}c^{-4}}$.

342. Obtain the cube roots of 51.064811, and $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

343. The prime cost of 38 gallons of wine is £25, and 8 gallons are lost by leakage: at what price per gallon should the remainder be sold, to gain 10 per cent. upon the outlay?

344. If $a : b :: c : d$, shew that

$$a^2\{(a^2 + d^2) - (b^2 + c^2)\} = (a^2 - b^2)(a^2 - c^2).$$

345. Expand $\{2a - 3\sqrt{ax}\}^{\frac{5}{3}}$ and $\{3a - 2\sqrt[3]{a^2x}\}^{-\frac{5}{2}}$, each to five terms.

346. From a company of 50 men, 5 are draughted off every night on guard: on how many different nights can a different selection be made? and on how many of these will two given soldiers be found upon guard?

347. (i) $\dfrac{a(a^2 + x^2)}{a + x} = ax + b^2$ (iii) $\left.\begin{array}{l} axy = c(bx + ay) \\ bxy = c(ax - by) \end{array}\right\}$

(ii) $5x - 11y^{\frac{1}{2}} + 13z^{\frac{1}{3}} = 22,\ 4x + 6y^{\frac{1}{2}} + 5z^{\frac{1}{3}} = 31,\ x - y^{\frac{1}{2}} + z^{\frac{1}{3}} = 2.$

348. A person, having to walk 10 miles, finds that, by increasing his speed half a mile an hour, he might reach his journey's end $16\frac{2}{3}$ minutes sooner than he otherwise would: what time will he take, if he only begin to quicken his pace halfway?

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349. Divide $(x^3 - 1)a^3 - (x^3 + x^2 - 2)a^2 + (4x^2 + 3x + 2)a - 3(x+1)$ by $(x - 1)a^2 - (x + 1)a + 3$.

350. Multiply $\sqrt[3]{a^{-\frac{1}{2}}} + \sqrt{(a^{\frac{1}{2}}c)^{\frac{1}{3}}}$ by $\sqrt{a^{-\frac{1}{3}}} - \sqrt[3]{(a^{\frac{1}{2}}c)^{\frac{1}{2}}}$.

351. If $x = \sqrt[3]{\{-\frac{1}{2}r + \sqrt{(\frac{1}{4}r^2 - \frac{1}{27}q^3)}\}}$, find the value of $x^6 + rx^3 + \frac{1}{27}q^3$.

352. Extract the square root of $\frac{9}{4}x^3 - 5x^{\frac{5}{2}}y^{\frac{1}{2}} + \frac{179}{45}x^2y - \frac{4}{3}x^{\frac{3}{2}}y^{\frac{3}{2}} + \frac{4}{25}xy^2$.

353. Add together $\dfrac{\frac{1}{2}(1 + \sqrt{5})x - 2}{x^2 - \frac{1}{2}(1 + \sqrt{5})x + 1}$ and $\dfrac{\frac{1}{2}(1 - \sqrt{5})x - 2}{x^2 - \frac{1}{2}(1 - \sqrt{5})x + 1}$.

354. Find the sum to n terms and *ad inf.* of the G. P., whose first two terms are the A. and H. means between 1 and 2.

355. What is the least number which is divisible by 7 and 11 with remainders 6 and 10 respectively?

356. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, making away at the rate of 8 miles an hour: how long will the chase last?

357. Expand $\{2a - 3\sqrt{ax}\}^{-\frac{5}{3}}$ and $\{3a - 2\sqrt[3]{a^2x}\}^{\frac{5}{2}}$, each to five terms.

358. In what scale will the common number 803 be expressed by 30203? What are the greatest and least common numbers that can be expressed with five digits in it?

359. (i) $\dfrac{a^2}{b + x} + \dfrac{a^2}{b - x} = c$ (ii) $\dfrac{a^2}{b + x} - \dfrac{a^2}{b - x} = c$

(iii) $\dfrac{x}{a} + \dfrac{y}{b} = 1 = \dfrac{x - a}{b} + \dfrac{y - b}{a}$

360. A, B, C reaped a field together in a certain time: A could have done it alone in $9\frac{7}{9}$ hrs more, B in half the time that A could, and C in an hour less than B. What time did it take them?

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361. Divide $\sqrt[12]{x^{10}y^9} - z\sqrt[6]{x^7y^5} - \frac{3}{2}x\sqrt[4]{{}^3y} + \frac{3}{2}x^3yz\sqrt{x^{-4}y^{-1}}$ by $\sqrt{xy} - \frac{3}{2}\sqrt[6]{x^4y^3}$.

362. The edges of three cubes are a, b, $a+b$; shew that the greatest : difference between it and the sum of the others $:: (a^{\frac{1}{2}}b^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{1}{2}})^2 : 3$.

363. Extract the square root of $x + 1 - 2\sqrt[4]{x}\,(1 + \sqrt{x} + 3\sqrt{x}$.

364. Simplify $\sqrt[3]{72} - 3\sqrt[3]{\frac{1}{3}}$ and $\sqrt{2ax^2} - \sqrt{2ax^2 - 4ax + 2a}$.

365. If $x = \frac{1}{2}(\sqrt{3} + 1)$ find the value of $4(x^3 - 2x^2) + 2x + 3$.

366. A's money with $\frac{1}{2}$ of B's would be $\frac{1}{3}$ as much again as before; and if 2*s* be taken from A's present sum and added to B's, the latter amount will be $\frac{1}{3}$ of the former. What had they each at first?

367. Find the value of $\sqrt[3]{x + 6x}$, and square the result.

368. If the difference of two fractions be mn^{-1}, shew that m times their sum $= n$ times the difference of their squares.

369. The first term of an A. P. is $n^2 - n + 1$, the common difference 2: find the sum of n terms, and thence shew that $1 = 1^3$, $3 + 5 = 2^3$, $7 + 9 + 11 = 3^3$, &c.

370. Find the area of a court 250 ft long by 200 ft broad, (i) by the senary, (ii) by the duodenary scale.

371. (i) $\dfrac{1}{ab - ax} + \dfrac{1}{bc - bx} = \dfrac{1}{ac - ax}$ (ii) $nx + \dfrac{b}{x} = na + ba^1$
(iii) $x^2 + y^2 = 2a^2$, $x + y : x - y :: m : n$.

372. A cistern has three pipes, A, B, and C: by A and B together it can be filled in 36′, and emptied by C in 45′, whereas, if A and C were opened together, it would be emptied in $1\frac{1}{2}$ hr: in what time would it be filled, by A, by B, or by all together?

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373. Find $\dfrac{1}{mn - mz} + \dfrac{1}{np - nz} + \dfrac{1}{mz - mp}$, when $z = \dfrac{n}{m}(m-n+p)$.

374. Multiply $max^{\frac{5}{3}} + (m-1)a^2x^{\frac{2}{3}} + (m-2)a^3x^{-\frac{1}{3}}$ by $a^{-1}\sqrt[3]{x^4} - \sqrt[3]{x}$.

375. Extract the square root of $1 + m^3 + 2(1-m^2)\sqrt{m} + 3m - m^2$.

376. Simplify $\frac{xy}{x-y} \pm \sqrt{\frac{x^2y^2}{(x-y)^2} + \frac{x^2y}{x-y} - \frac{xy}{x \pm \sqrt{xy}}}$.

377. Find a number of two digits such that its quotient by their sum exceeds the first digit by 1, and equals the other.

378. How many terms of the series $-7-5-3-$ &c. amonnt to 9200? and how many of $6+4+2\frac{2}{3}+$ &c. amount to $14\frac{4}{9}$?

379. A certain number of men mowed 4 acres of grass in 3 hours, and a certain number of others mow 8 acres in 5 hours: how long would they be in mowing 11 acres, all working together?

380. If a, b, c, d are in G. P., shew that

$$(a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(b+c)^2.$$

381. The No. of Var^{ns} of n things, r together : the No., $r-1$ together :: 10 : 1, and the corresponding Nos. of Comb^{ns} are as 5 : 3; find n and r.

382. A person makes 20 lbs. of tea at 4*s* 9*d*, by mixing three kinds at 3*s* 6*d*, 4*s* 6*d*, and 5*s*: how can this be done?

383. (i) $\frac{1}{2}(x - 1\frac{25}{26}) - \frac{1-3x}{6\frac{1}{2}} = x - \frac{1}{39}\{5x - \frac{5}{2}(1 - .3x)\}$

(ii) $x+a+b+c = \frac{x^2+a^2+b^2+c^2}{a+b+c+x}$ (iii) $\left.\begin{aligned}\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= 2\\ ay+bx &= 0\end{aligned}\right\}$

384. A trader maintained himself for 3 years at an expense of £50 a year, and in each of these years increased that part of his stock which was not so expended by $\frac{1}{3}$ thereof: at the end of 3 years his original stock was doubled: find it.

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385. Divide $(6a^2-7ab+2b^2)\,x^3+(5a^3-3a^2b-5ab^2+3b^3)\,x^2+(a^2-b^2)\,{}^2x$ by $(2a-b)\,x + a^2 - b^2$.

386. Find the L. C. M. of

$$x^4 - (p^2+1)\,x^2 + p^2 \text{ and } x^4 - (p+1)^2\,x^2 + 2\,(p+1)\,px - p^2.$$

387. Obtain the values of (i) $x - \sqrt{xy} + y$, and (ii) of x^2+xy+y^2, when $x = \frac{1}{16}(4\frac{1}{8} + \sqrt{7\frac{2}{3}})$, $y = \frac{1}{16}(4\frac{1}{3} - \sqrt{7\frac{2}{3}})$.

388. Simplify $(a-b\left\{\frac{1}{(x+a^2)} + \frac{1}{(x+b)^2} + 2\left\{\frac{1}{x+a} - \frac{1}{x+b}\right\}\right.$.

389. Obtain the square roots of

$$2 + a^{2\sqrt{2}} + a^{-2\sqrt{2}} \text{ and } \frac{a^3c}{b} + cf - 2ac\sqrt{\frac{f}{b}}.$$

390. The n^{th} term of an A. P. is $\frac{1}{2}n - \frac{1}{6}$: find the sum of n terms.

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391. The diagonal of a cube is a foot longer than each of the sides: find the solid content.

392. Find the first time after noon when the hour and minute hands of a watch point exactly in opposite directions.

393. In how many ways may £10 be paid in crowns, sevenshilling pieces, and moidores (27*s*), thirty coins being used?

394. Out of 5 white, 7 red, and 8 black balls, how many different sets of 6 balls could be drawn, (i) two of each color, (ii) one white, two red, three black, (iii) three red, three black?

395. (i) $x + \sqrt{x^2 - 2ax + b^2} = a + b$

(ii) $\dfrac{a}{x+a} - \dfrac{c}{x-c} = \dfrac{a-c}{x+x-c}$ (iii) $\sqrt{1 + \dfrac{bx}{a^2}} + \sqrt{1 - \dfrac{bx}{a^2}} = 1\frac{3}{5}$.

396. Two vessels, *A* and *B*, contain each a mixture of water and wine, *A* in the ratio of 2 : 3, *B* in that of 3 : 7. What quantity must be taken from each, to form a mixture which shall consist of 5 gallons of water and 11 of wine?

397. Shew that $(ay - bx)^2 + (cx - az)^2 + (bz - cy)^2$

$$= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2.$$

398. Find the G. C. M. of $3x^2 + (4a - 2b)x - 2ab + a^2$ and

$$x^3 + (2a - b)x^2 - (2ab - a^2)x - a^2b.$$

399. From $\frac{1}{2}(x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})(x^{\frac{1}{2}} - 2x^{-\frac{1}{2}})$ take $\frac{1}{3}(x^{\frac{1}{2}} + 2x^{-\frac{1}{2}})(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$, and multiply the result by $6(1 - x^{-1})^{-1}$.

400. Extract the square root of $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} - 2x^{-\frac{1}{6}} + x^{-\frac{1}{3}} - 1$.

401. Multiply together

$$\sqrt[2n]{(a+b)^{m+1}},\ \sqrt{(a+b)^{\frac{n+1}{m}}},\ \sqrt{(a+b)^{\frac{m-1}{n}}},\ \sqrt[m]{(a+b)^{\frac{n-1}{2}}}.$$

402. Simplify

$$\left\{\frac{1+x}{1-x} + \frac{4x}{1+x^2} + \frac{8x}{1+x^4} - \frac{1-x}{1+x}\right\} \div \left\{\frac{1+x^2}{1-x^2} + \frac{4x^2}{1+x^4} - \frac{1-x^2}{1+x^2}\right\}$$

403. Sum $(a+x)^2 + (a^2+x^2) + (a-x)^2 +$ &c. to 5 and to n terms.

404. Find two numbers such that their sum, product, and difference of their squares may be equal.

405. Apply the Bin. Theor. to find $(1.01)^{-\frac{3}{2}}$ to nine places.

406. Find the least integer which, when divided by 7, 8, 9, respectively, shall leave remainders 5, 7, 8.

407. (i) $x + 3 = \sqrt{2(x+3)} + 4$ (ii) $abx^2 - (a+b)cx + c^2 = 0$

(iii) $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1,\ \dfrac{x}{b} - \dfrac{y}{a} = 0$

408. A person bought 38 sheep for £57; but, having lost a certain number, n, of them, he sold the remainder for n shillings a head more than they cost him, and so gained upon the whole 16*s*: how many sheep did he lose?

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409. Shew that $(a^2 + b^2 - 1)^2 + (a'^2 + b'^2 - 1)^2 + 2(aa' + bb')^2$
$= (a^2 + a'^2 - 1)^2 + (b^2 + b'^2 - 1)^2 + 2(ab + a'b')^2$.

410. Find the G. C. M. of $xy + 2x^2 - 3y^2 + 4yz + xz - z^2$ and
$2x^2 - 9xz - 5xy + 4z^2 + 8yz - 12y^2$.

411. Find the fourth term of $(\sqrt{2} + \sqrt{3})^6$, correct to four places.

412. Obtain the square root of $1 + x - \frac{3}{2}\sqrt[3]{x}(1 + \sqrt{x}) + \sqrt{x}(2 + \frac{9}{16}\sqrt[6]{x})$.

413. If the r^{th} term of a series be ar^{1-r}, shew that the sum of the m^{th} and n^{th} terms exceeds the $(m+n)^{th}$ by $\frac{m^2+mn+n^2}{mn(m+n)}a$.

414. If $x^{-1} = (a-c)(b-c)$, $y^{-1} = (a-b)(b-c)$, $z^{-1} = (a-b)(a-c)$, find the values of $x - y + z$ and $abx - acy + bcz$.

415. If P, Q, R, be the p^{th}, q^{th}, and r^{th} terms of any H. P., shew that $(p-q)PQ + (q-r)QR + (r-p)RP = 0$.

416. Two parcels of cotton, weighing 9 lbs and 16 lbs, cost 11*s* 6*d* and £1 0*s* 4*d* respectively, and the charge for carriage was proportional to the square root of the weight: how much per lb was paid for the purchase of the cotton?

417. If $a : b :: b : c$, shew that $a + b : b + c :: a^2(b-c) : b^2(a-b)$.

418. Find the least number which, being divided by 2, 3, 5, shall leave remainders 1, 2, 3.

419. (i) $(x-1) + 2(x-2) + 3(x-3) +$ &c. to six terms $= 1$ 4

(ii) $\frac{2x(a-x)}{3a-2x} = \frac{1}{4}a$. (iii) $\frac{x}{a} + \frac{y}{b} = 1 = \frac{x}{a} + \frac{z}{c}$, $yz = bc$.

420. A square court-yard has a rectangular walk around it; the side of the court wants 2 yds. of being six times the breadth of the walk, and the No. of sq. yds. in the walk exceeds by 92 the No. of yds. in the periphery of the court: find its area.

ANSWERS TO THE EXAMPLES.

1. 1. 48. 2. 12. 3. -8. 4. 1 5. 106.
6. -1. 7. -178. 8. 150. 9. 450. 10. 192.

2. 1. 11. 2. 1. 3. 0. 4. 94. 5. 89.
6. -64. 7. 16. 8. 264. 9. 5. 10. 3.

3. 1. 25. 2. -15. 3. 12. 4. 6. 5. 21.
6. 22. 7. 7. 8. 13. 9. 15. 10. 4.

4. 1. 46. 2. 24. 3. 35. 4. 10. 5. 7200.
6. 135. 7. 8. 8. 120. 9. 384. 10. 4.

5. 1. $15a + 3b - 6c + 6d$. 2. $14x - 9y + 10z - 12$.
3. $23a^2 - 26ab + 14b^2$. 4. $6by - 7cz$.
5. $5x^3 + 50x^2y - 14xy^2 + 4y^3$. 6. $2x^2 + 2y^2 + 2z^2$.
7. $-9x^3 + 2ax^2 - 31a^2x + 16a^3$. 8. $a^3 + b^3 + c^3 + 6abc$.
9. $6x^3 + 4y^3 + z^3 - 24xyz$. 10. $x^4 + y^4 + z^4$.

6. 1. $a - 3b + 3c$. 2. $-2x^2 - 7xy + 3y^2$. 3. $4ax - 9by + 2cz$.
4. $5x^2 - 5x + 5$. 5. $7a^3 - 3a + 4b^2 - 7ab + 2c^2 - 6bc$.
6. $-x^3 - 6x^2y - 2y^2 + 6 - 3x^2 - 4y^2$.
7. $3x^2 + 13xy - y^2 - 16xz - 13yz$. 8. $x^2 + xy + y^2$.
9. $3a^4 - 4a^3b - 4ab^3 + 2b^4$. 10. 0.

7. 1. $4a - 4x$. 2. $4a^3 - 4a^2c$. 3. $x^2 - 3y^2 - 3z^2$.
4. $2ax^2 + 2by^2 + 2cz^2$. 5. $a^2 - 3b^2 + 3c^2$. 6. $2ab + 4b^2$.
7. 0. 8. $-3x - y + 4z$. 9. $8x - 8$. 10. $-4c + 4d$.

8. 1. $(2a - b) - (3c - 4d) - (2e - 3f)$, $(2a - b - 3c) + (4d - 2e + 3f)$.
2. $-(b + 3c) + (4d - 2e) + 3f + a)$, $-(b + 3c - 4d) - (2e - 3f - a)$.
3. $-(3c - 4d) - (2e - 3f) + (2a - b)$, $-(3c - 4d + 2e) + (3f + 2a - b)$.
4. $(4d - 2e) + (3f + 2a) - (b + 3c)$, $4d - 2e + 3f) + (2a - b - 3c)$.
5. $-(2e - 3f) + (2a - b) - (3c - 4d)$, $-(2e - 3f - 2a) - (b + 3c - 4d)$.
6. $(3f + 2a) - (b + 3c) + (4d - 2e)$, $(3f + 2a - b) - (3c - 4d + 2e)$.
7. $\{2a - (b + 3c)\} + \{4d - (2e - 3f)\}$. 8. $-\{b + (3c - 4d)\} - \{2e - (3f + a)\}$.
9. $-\{3c - (4d - 2e)\} + \{3f + (2a - b)\}$. 10. $\{4d - (2e - 3f)\} + \{2a - (b + 3c)^3$.
11. $-\{2e - (3f + 2a)\} - \{b + (3c - 4d)\}$. 12. $\{3f + (2a - b)\} - \{3c - (4d - 2e)\}$.

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9. 1. $(a - b + c)x^3 - (b - c + d)x^2 - (c + d + e)x$. 2. $2(ax - by)$.
3. $(a + b)x^2 - (a - 5b)xy + (a - c)y^2$. 4. $2(ax + cy)$, $2b(x + y)$.
5. $-(a - 5b)x + (2a + 3b + c)y$, $(a - 4b - c)x + (a - 3b - 2c)y$, $(b - c)x + (3a - c)y$.
6. $(5a - b)x - (2a - 3b - 5c)y$, $-(a + c)x + (a - b + 2c)y$, $(4a - b - c)x - (a - 2b - 7c)y$.
7. $(2a + 4b + c)x - (a - 5b - 3c)y$, $-(4a - 5b)x + (2a + b)y$, $-(2a - 9b - c)x + (a + 6b + 3c)y$.
8. $(a + 4b)x + (4b + 5c)y$, $-(3a - 5b - c)x + (a + 2b - 2c)y$, $(2a - 9b - c)x + (a + 6b + 3c)y$.

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10. 1. abx^3y^4, $-mnx^5$, $2a^2cx^2y$, ab^2c^2, a^2bc^2, $-x^3y^3$.
2. $x^3 - x^2y + xy^2$, $-a^3x + a^2x^2 - ax^3$, $-abx^3 + a^2bx^2 - ab^2x$, $x^4y - 3x^3y^2 + 3x^2y^3 - xy^4$.
3. $2a^2 + 7ab + 3b^2$, $2ac - bc - 6ad + 3bd$.
4. $6x^2 + 13xy + 6y^2$, $6a^2b^2 - ab^3 - 12b^4$.
5. $x^3 + 6x^2 + 7x - 6$, $x^3 - 6x^2 + 11x - 6$.
6. $a^4 + a^2 - 2a^2 + 3a - 1$, $a^4 - a^3 - 8a^2 + a + 1$.
7. $81x^4 - y^4$. 8. $a^5 + 32b^5$. 9. $x^4 - 4a^3x + 3a^4$.
10. $27a^3 + b^3 + 8 - 18ab$. 11. $x^3 - y^3 + z^3 + 3xyz$. 12. $a^6 - 1$.
13. $a^3 - 8b^3 - 27c^3 - 18abc$. 14. $a^6 + 2a^3b^3 + b^6$.

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15. $x^3 - (a + c)x^2 + (ac + b)x - bc$; $x^4 - (a^2 - b + c)x^2 + a(b + c)x - bc$.
16. $1 - (a - 1)x - (a - b + 1)x^2 + (a + b - c)x^3 - (b + c)x^4 + cx^5$.
17. $a^2 - amx - 2m^2x^2 + 3mnx^3 - n^2x^3$; $a^2 + a(m + 2n)x - \{a(m + n) - 2mn\}x^2 - (m^2 + 2n^2)x^3 + mnx^4$.
18. $a^3x^3 - a^2(b - c + d)x^2y - (abc - abd + acd)xy^2 + bcdy^3$.
19. $4x^4 + 6(m - n)x^3 - (4m^2 + 9mn + 4n^2)x^2 + 6mn(m - n)y + 4m^2n^2$.
20. $x^5 - (2a^2 + 2b^2 + ab)x^3 + (a^4 + a^3b + a^2b^2 + ab^3 + b^4)x - (a + b)a^2b^2$

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11. 1. $a^2-2ax+x^2$, $1+4x^2+4x^4$, $4a^4+12a^2+9$, $9x^2-24xy+16y^2$.
2. $9+12x+4x^2$, $4x^2-12xy+9y^2$, $a^4-6a^3x+9a^2x^2$, $b^2x^4-2bcx^3y+c^2x^2y^2$.
3. $4a^2-1$, $9a^2x^2-b^2$, x^4-1.
4. x^2+4x+3, x^4+3x^2-4, a^2b^2-ab-6, $4a^2x^2-8abx+3b^2$.
5. $x^4-5a^2x^2+4a^4$.
6. $m^4x^4-13m^2n^2x^2y^2+36n^4y^4$.
7. $4x^2$.
8. x^4+4y^4, $4a^4-5a^2b^2+b^4$.
9. $a^2+2ab+b^2-c^2$, $a^2-b^2+2ac+c^2$, $a^2-b^2-2bc-c^2$.
10. $a^2-2ab+b^2-c^2$, $-a^2+2ab-b^2+c^2$, $-a^2+b^2-2bc+c^2$.
11. $4a^2-b^2+6bc-9c^2$, $-4a^2+12ac+b^2-9c^2$.
12. $4a^2-b^2-6bc-9c^2$, $-4a^2+4ab-b^2+9c^2$.
13. $a^2+2ac+c^2-b^2-2bd-d^2$, $a^2+2ad+d^2-b^2-2bc-c^2$, $b^2+2bc+c^2-a^2-2ad-d^2$.
14. $a^2+2ad+d^2-4b^2+12bc-9c^2$, $9c^2+6cd+d^2-a^2+4ab-4b^2$, $a^2+6ac+9c^2-4b^2+4bd-d^2$.

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12. 1. bc^2, $5xy^2$, $-35bx$.
2. $3xy-2xz+3yz$, $-a^2b^2+7abc^2-4c^4$, $\frac{a^2x^2}{b}-3ax+3by-\frac{b^2y^2}{a}$.
3. $-\frac{2m}{3n}+1-\frac{4m^2}{3n^2}+\frac{n}{3m}$, $\frac{3a^2}{2b^2}-\frac{5a}{2b}+3+\frac{b}{2a}-\frac{2b^2}{a^2}$.
4. $x+5$, m^2-4m+3.
5. $3a-2b$, $3x+2y$.
6. $2ab-3b^2$.
7. $a^2-2ab+2b^2$, $2x^2y^2+2xy+1$.
8. $x^4-2x^3y+4x^2y^2-8xy^3+16y^4$.
9. $1-2x+3x^2-4x^3+5x^4$.
10. $x^2+2xy+3y^2$, m^2-2m+3.
11. $a^3+2a^2b+3ab^2+4b^3$.
12. $x^4+2x^3+3x^2+2x+1$, $a^4-2a^3b+3a^2b^2-2ab^3+b^4$.

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13. 1. x^2-px+q.
2. az^2+bz-c.
3. $y^4-(m-1)y^3-(m-n-1)y^2-(m-1)y+1$.
4. $a+b-c-d$.
5. $a+2b-c$.
6. $a^2+b^2+c^2+ab-ac+bc$, $a^2+b^2+c^2+ab+ac-bc$.
7. $1-x+2y+x^2+2xy+4y^2$, $1+x-2y+x^2+2xy+4y^2$.
8. $x^2+4y^2+9z^2+2xy+3xz-6yz$.
9. $x^2+y^2+z^2+1$.
10. $a-ax+ax^2-ax^3+\frac{ax^4}{1+x}$, $1+5x+15x^2+45x^3+\frac{135x^4}{1-3x}$.
11. $1+2x+3x^2+4x^3+\frac{5x^4-4x^5}{1-2x+x^2}$,

$$1-(a+b)x+(a+b)bx^2-(a+b)b^2x^3+\frac{(a+b)b^3x^4}{1+bx}.$$

12. a^3-pa^2+qa-r.

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14. 1. $a-x$, $a^4+a^3x+a^2x^2+ax^3+x^4$, $a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5$.

2. $3x+1$, $5x-1$, $2x-3$. 3. $3mn-5$, $4m^2-n^3$.

4. $1-2x+4x^2$, $9x^2+3x+1$, $1-2x+4x^2-8x^3$.

5. $x^3+3x^2y+9xy^2+27y^3$, $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$, $x^{15}-x^{12}y^2+x^9y^4-x^6y^6+x^3y^8-y^{10}$.

6. $\frac{1}{4}a^2-\frac{1}{2}ab+b^2$, $x^3y^3-x^2y^2z+xyz^2-z^3$. 7. $a+b+c$, $a+b-c$.

8. $(x+y)^2-(x+y)z+z^2=x^2+2xy+y^2-xz-yz+z^2$, $x^2+x(y-z)+(y-z)^2=x^2+xy-xz+y^2-2yz+z^2$.

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15. 1. $(1-2x)(1+2x)$, $(a-3x)(a+3x)$, $(3m-2n)(3m+2n)$, $x^2(5a-2)(5a+2)$, $x^2y^2(4x-5y)(4x+5y)$.

2. $(x+y)(x^2-xy+y^2)$, $(x-y)(x^2+xy+y^2)$, $(1+xy)(1-xy+x^2y^2)$, $(x-1)(x+1)(x^2+1)$, $xy(ay-x^2)(ay+x^2)$, $2ab^2c(a-2c)(a+2c)$.

3. $x^3(5x-a)(5x+a)$, $a^4(a-3b^3)(a+3b^3)$, $(2x-3)(4x^2+6x+9)$, $(a-2b)(a^2+2ab+4b^2)$, $x^2y(a+3y)(a^2-3ay+9y^2)$.

4. $(x+2)(x^4-2x^3+4x^2-8x+16)$, $x^3(a+3x)(a^2-3ax+9x^2)$, $(2x^3+y^2)(4x^6-2x^3y^2+y^4)$, $(ab^3+c^2)(ab^3-c^2)(a^2b^6+c^4)$, $abc(a+c)^2$.

5. $(3x-1)(3x+1)(9x^2+1)$, $(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$, $x^2(x-b)^2$, $x^2(x-a)^2(x+a)^2$.

6. $(4x-5)(2x+1)$, $(a+3b)(a-b)$, $7(x-y)(x+y)$.

7. $(x-y)^2(x+y)^2$, $(c+a-b)(c-a+b)$, $8ab$.

8. $(x+y)^3$, $mn(m-n)$, $5b(a-b)$.

9. $2(x+y)(4x-y)$, $2(x-y)(4y-x)$, $4y(x+y)$.

10. $(a+b)(a^2+ab+b^2)$, $(a-b)^3$, 0.

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16. 1. $(x+1)(x+5)$, $(x+4)(x+5)$, $(x-2)(x-3)$, $(x-3)(x-5)$, $(x+1)(x+7)$, $(x-1)(x-9)$.

2. $(x+3)(x-2)$, $(x-3)(x+2)$, $(x-3)(x+1)$, $(x+5)(x-3)$, $(x+8)(x-1)$, $(x-9)(x+1)$.

3. $(2x+3)(2x+1)$, $(4x+1)(x+3)$, $(4x-1)(x+3)$, $(2x-3)(2x+1)$, $(3x-2)(x+2)$, $(3x+4)(2x-1)$.

4. $(4x+1)(3x-2)$, $2(6x-1)(x-1)$, $(4x+1)(3x-1)$, $(x+4)(x-3)$, $(3x-5)(x+1)$.

5. $a^2(x-a)(x-2a)$, $a(a-3x)(a+2x)$, $ab(3a-2b)(a+b)$, $(4a^2-x^2)(3a^2+x^2)$.

6. $xy(2x+y)(x+2y)$, $3y^2(3x+2y)(x-y)$, $a^2(3ax-1)(2ax+1)$, $x^2(2b-3x)(3b+x)$.

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17. 1. 5. 2. 2. 3. 3. 4. $\frac{4}{7}$. 5. $-\frac{1}{2}$.
6. $\dfrac{d-a}{m-n}$. 7. 2. 8. 1. 9. 4. 10. $-\frac{1}{9}a$.
11. -4. 12. $\frac{5}{6}$. 13. $-\frac{2}{3}$ 14. $\dfrac{m^2}{n}$.

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18. 1. 42. 2. 12. 3. 12. 4. 5. 5. 7.
6. 4. 7. 5. 8. $\frac{2}{3}$. 9. 7.
10. $\frac{1}{13}(25a-18b)$. 11. 7. 12. -8.

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19. 1. 4. 2. 2. 3. 18. 4. 8. 5. $-a$. 6. 6.
7. 4. 8. $b-a$. 9. 7. 10. $a-m$. 11. 10. 12. $2(a+c)$.

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20. 1. 12. 2. 9. 3. 120. 4. 7*s*. 5. 35, 13.
6. 513, 466. 7. 15. 8. 31, 18. 9. 15. 10. 90, 60.
11. 24 ft. 12. 16. 13. 37, 30, 20. 14. 20. 15. 41.
16. £5. 17. 88. 18. 85*s*, 35*s*. 19. £36, £12, £16.
20. 5. 21. £45, £57, £63, £65. 22. 15, 5.
23. $98\frac{2}{3}$ miles from L, $10\frac{2}{3}$ h. 24. 22, 7, 12 gals.
25. 1 h 20′ from *B*'s starting, $6\frac{1}{3}$ miles. 26. 3000.
27. 3*s*, 5*s*, 7*s*. 28. £189. 29. 8. 30. 25.

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21. 1. $4a^2b^4$, $-27a^6b^6c^{12}$, $\dfrac{81a^4b^8}{256c^{12}}$, $-\dfrac{x^{10}y^{15}z^{20}}{32}$.

2. $x^3+6x^2+12x+8$. 3. $x^4-8x^3+24x^2-32x+16$.

4. $x^5+15x^4+90x^3+270x^2+405x+243$.

5. $1+10x+40x^2+80x^3+80x^4+32x^5$.

6. $8m^3-12m^2+6m-1$. 7. $81x^4+108x^3+54x^2+12x+1$.

8. $16x^4-32ax^3+24a^2x^2-8a^3x+a^4$.

9. $243x^5+810ax^4+1080a^2x^3+720a^3x^2+240a^4x+32a^5$.

10. $64a^3-144a^2b+108ab^2-27b^3$.

11. $a^3x^3-3a^2x^2y^2+3axy^4-y^6$.

12. $a^4x^4+4a^3x^5+6a^2x^6+4ax^7+x^8$.

13. $32a^5m^5-80a^4m^6+80a^3m^7-40a^2m^8+10am^9-m^{10}$.

14. $a^3-3a^2b+3a^2c+3ab^2-6abc+3ac^2-b^3+3b^2c-3bc^2+c^3$.

15. $1-3x+6x^2-7x^3+6x^4-3x^5+x^6$.

16. $a^3+3a^2bx+3a(b^2+ac)x^2+(6ac+b^2)bx^3+3(ac+b^2)cx^4+3bc^2x^5$
$+c^3x^6$.

17. $1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$.

18. $1+5x+5x^2-10x^3-15x^4+11x^5+15x^6-10x^7-5x^8+5x^9-x^{10}$.

19. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

20. $a^4-8a^3b+4a^3c+24a^2b^2-24a^2bc+6a^2c^2-32ab^3+48ab^2c-24abc^2$ $+4ac^3+16b^4-32b^3c+24b^2c^2-8bc^3+c^4$.

21. $1+10x+25x^2-40x^3-190x^4+92x^5+570x^6-360x^7-675x^8$ $+810x^9-243x^{10}$.

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22. 1. $1+2x+3x^2+2x^3+x^4$. 2. $1-2x+5x^2-4x^3+4x^4$.
3. $9-12x+10x^2-4x^3+x^4$. 4. $a^4-4a^3b+10a^2b^2-12ab^3+9b^4$.
5. $4x^2+9y^2+16z^2-12xy+16xz-24yz$.
6. $9a^2x^2+4b^2y^2+c^2z^2+12abxy+6acxz+4bcyz$.
7. $1-4ax+2a^2x^2+4a^3x^3+a^4x^4$.
8. $4a^4-4a^3-7a^2+4a+4$.
9. $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$.
10. $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.
11. $x^6-4x^5+10x^4-4x^3-7x^2+24x+16$.
12. $1+4x-2x^2-4x^3+25x^4-24x^5+16x^6$.
13. $a^6-4a^5b+8a^4b^2-10a^3b^3+8a^2b^4-4ab^5+b^6$.
14. $a^8-8a^7x+28a^6x^2-56a^5x^3+70a^4x^4-56a^3x^5+28a^2x^6-8ax^7+x^8$.
15. $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$.
16. $a^8-4a^7x+6a^6x^2-8a^5x^3+11a^4x^4-8a^3x^5+6a^2x^6-4ax^7+x^8$.

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23. 1. $\pm 2ab^2c^3$, $\pm 7x^2y^3z$, $\pm 10a^4b^6c^3$.
2. $\pm \frac{3ax^2y^3}{5z}$, $\pm \frac{7xy^2}{8a}$, $\pm \frac{5x^3y^5}{4ab^2}$.
3. $\pm \frac{a^2x^2y}{2}$, $-\frac{2ay^2}{3x^3}$, $\frac{4b^2c^3}{5a^4}$, $-\frac{6abc^5}{7}$.
4. $\pm \frac{2xy^2}{5a^3}$, $\pm \frac{3ab^2c^3}{4x^4}$, $\frac{2ab^2}{c^3}$, $\pm \frac{2x^2y}{3z^3}$.

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24. 1. $2x+y$, $5a-3b$, $5x^2+3xy$.
2. $7ab-a^2$, $4xy+5yz$, $5a^2bc+c^4$.

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25. 1. $1+2x+3x^2$. 2. $3x^2+2x+3$. 3. $3a+2b+c$.
4. $x^2-4xy+4y^2$. 5. $2a^3-3a+4$. 6. $4x^2-2ab+2b^2$.
7. x^3-2x^2+3x-4. 8. $3a-b+5c+d$. 9. $x^3-2x^2y+2xy^2-y^3$.
10. $1-3x+3x^2-x^3$. 11. $2-3a-a^2+2a^3$.
12. $p+qx+rx^2+sx^3$. 13. $1-x$, $a-2$.
14. $2a-3b$. 15. $x-2$. 16. $a-b$.

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26. 1. 421, 347, 69.4, 737, 1046, 4321.
2. 2082, 20.92, 1011, 20.22, 129.63.
3. 3789, 75.78. 15.156, 8642, 2211.
4. 4.164, 8328, 2568.2, 11367, 31230.
5. 4.044, 8088, 5055, 6633, 15165.
6. 1.5811, 44.721, .54772, .17320, 10.535, .03331, .06324, .07071.

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27. 1. $x + 2y$. 2. $a - 3$. 3. $x + 4$. 4. $2a - 3b$.
5. $a + 8b$. 6. $2x - 7y$. 7. $m - 4nx$. 8. $ax - 5bx$.

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28. 1. $a^2 + 2a + 1$. 2. $x^2 - 4x + 2$. 3. $a^2 - ab + b^2$.
4. $x^2 - 4ax + 4a^2$. 5. $2x^2 + 4xy - 3y^2$. 6. $x^3 - x^2 + x - 1$.
7. $a - b + c$. 8. $1 - 2x + 3x^2 - 4x^3$.

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29. 1. 21, 23, 25, 32, 4.7, 48, 64, 9.6.
2. 114, 11.7, 125, 108, 1.41, 192.
3. 2.34, 206, 3.84, 32.1, 282.
4. 46.8, 936, 6.42, 1025, 1.284. 5. 1.357, .5848, .2154, 1.587.

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30. 1. $3x^2$, $2ab^2$, $4y^2z^2$. 2. ax, a, x.

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31. 1. $2x^2(a+x)^2$. 2. $x^2(a+x)^2$. 3. $ab(a-b)^2$. 4. $2(x-1)$.
5. $x^2(x+1)$. 6. $2(x+a)$. 7. $a^2(x+1)$. 8. $3(ax+2)$.

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32. 1. $3x - 2$. 2. $2x + 3$. 3. $3x + 5$.
4. $8x^2 + 14x - 15$. 5. $4x - 5$. 6. $x^2 + 2x - 3$.

---

33. 1. $a + x$. 2. $x - 1$. 3. $2(x^2 + 2x + 1)$.
4. $y - 2$. 5. $x - 2a$. 6. $x + 3$.
7. $3(x + 3)$. 8. $x^2 + y^2$. 9. $a(a + b)$.
10. $a(a^3 - b^2)$. 11. $x^2 - 2xy + y^2$. 12. $x^2 + 4x + 4$.

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34. 1. $2x + 3$. 2. $3x - 2$. 3. $3x - 2$.
4. $x - 1$. 5. $x - 3$. 6. $x - y$.
7. $x + 3$. 8. $x^2 - 3$. 9. $5x^2 - 1$.
10. $3x^2 - 2xy + y^2$. 11. $x(2x^2 + 2xy - y^2)$. 12. $x - 1$.

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35. 1. $12a^2b^2c$, $36x^3y^3$, $ax^2y - axy^2$, $ab^3 - ad^2$.
2. $120a^4b^2$, $10a^5b^5$, $1800a^3x^3$.
3. $6(a^2 - b^2)$, $12a(a^2 - 1)$, $120xy(x^2 - y^2)$.
4. $24a^2b^2(a^2-b^2)$, $36xy^2(x^2-y^2)$.

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36. 1. $\frac{a+y}{a}$, $\frac{x}{a^2}$, $\frac{m}{3(m-2x)}$, $\frac{7x}{5a}$, $\frac{a^2-3ab}{2b(a+2b)}$.

2. $\frac{x^2-3y^2}{y(x-2y)}$, $\frac{2mn}{m+n}$, $\frac{3abc}{a+b+c}$, $\frac{3xy-5y^2}{4x-7y}$.

3. $\frac{c}{2df}$, $\frac{c+y}{f+2x}$, $\frac{cx+d}{ax+b}$.

4. $\frac{x-1}{a}$, $\frac{x^2+a^2}{x^3}$, $\frac{a^4+a^2b^2+b^4}{a^2+b^2}$, $\frac{x^2-bx}{x+b}$, $\frac{a-b}{a+b}$.

5. $\frac{x-1}{x+1}$. 6. $\frac{x-1}{x+2}$. 7. $\frac{a+b}{a-b}$. 8. $\frac{3a-2x}{5a+3x}$.

9. $\frac{2a-3x}{2a+3x}$. 10. $\frac{x+4}{x^2-2x+1}$. 11. $\frac{7x-2y}{5x^2-3xy+2y^2}$.

12. $\frac{5a^3(a+x)}{x(a^2+ax+x^2)}$. 13. $\frac{x^2+4x+4}{x^2+x+1}$. 14. $\frac{x^2+x-2}{x^2+5x+5}$.

15. $\frac{x^2-ax+a^2}{x^2-a^2}$. 16. $\frac{3ax^2+1}{4a^2x^4+2ax^2-1}$.

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37. 1. $3x-6+\frac{29}{x+4}$, $a-2x+\frac{3x^2}{a+x}$, $2x+6+\frac{23}{x-3}$

$2a-3x+\frac{7x^2}{5a-x}$, $12x+3+\frac{19}{4x-1}$.

2. $\frac{x(x^2-2x-3)}{x-2}$, $\frac{a^3+2x^3}{a+2x}$, $\frac{x^3+xy+y^3}{x+a}$.

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38. 1. $\frac{bcx,\ acy,\ abz}{abc}$, $\frac{6cx^2,\ 4by^2,\ 3az^2}{12abc}$.

$\frac{40b^3x^2y,\ 45ab^2x^3,\ 48a^2by^3,\ 50a^3xy^2}{60a^3b^3}$.

2. $\frac{a^2x^2-b^2x^2,\ a^2y^2+b^2y^2}{a^4-b^4}$, $\frac{a^2+2ax+x^2,\ a^2-2ax+x^2}{a^2-x^2}$,

$\frac{8ax^2-8bx^2,\ xy}{6(a^2-b^2)}$. 3. $\frac{a-x,\ a+x,\ 2a}{4a^3(a^2-x^2)}$.

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39. 1. $\frac{a^2+b^2}{2(a+b)b}$, $\frac{3a^2-ab+2b^2}{6(a-b)b}$, $\frac{25a-20b}{12}$.

2. $\frac{ab}{a-b}$, $\frac{a^2+b^2}{a^2-b^2}$, $\frac{a^2+b^2}{a^2-b^2}$, $\frac{a^2-ab+b^2}{a^2-b^2}$.

3. $\frac{a+bx}{c+dx}$, $\frac{2a^2-2ab+2b^2}{a^2-b^2}$, $\frac{2ab}{a^2-b^2}$, $\frac{x-y}{x}$.

4. $\dfrac{2a^3}{a^4 - x^4}$, 0. 5. $\dfrac{1}{x^2(x^2-1)}$. 6. $\dfrac{a}{4a^2 - b^2}$. 7. $\dfrac{a + bx}{b + ax}$.

8. $\dfrac{1 + x^3 + x^5}{x^2(x^2+1)^2}$. 9. $\dfrac{2x}{x+y}$. 10. $\dfrac{4x^3y - x^2y^2 - y^4}{x^4 - y^4}$.

11. $\dfrac{a^2 + x^2}{a^2(a+x)}$. 12. $\dfrac{2x^4 + 4x^2y^2 - 2y^4}{x^4 - y^4}$. 13. $\dfrac{a^2 + x^2}{a^2(x-a)}$.

14. $\dfrac{y}{x+y}$. 15. $\dfrac{x - 3x^2 + 3x^3}{(1-x)^3}$. 16. $\dfrac{1 + 2x + 3x^2}{4(1-x^4)}$

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40. 1. $9ax$, $\dfrac{ax + x^2}{b(a-x)}$, 1, $\dfrac{3a^2(a-b)}{b}$.

2. $\dfrac{a^4 - x^4}{a^2x}$, $\dfrac{a^3 + 2a^2x + 2ax^2 + x^3}{(a-x)(a^2 - ax + x^2)}$, $\dfrac{2ax^2(x-y)}{c}$.

3. $\dfrac{ab}{a^2 + 4b^2}$, $\dfrac{x}{x^2 + y^2}$, $\dfrac{a^4x(ax-1)}{a-b}$. 4. $\dfrac{a^2 + b^2}{(a-b)^2}$, 1, $\dfrac{x-1}{x^2}$

5. $\dfrac{(a^2 - b^2)b}{a^3}$, $\dfrac{3a}{2b}$. 6. $\dfrac{x^2 + b^2}{x - b}$.

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41. 1. $\dfrac{4-3x}{10}$, $\dfrac{3x}{15-2x}$, $\dfrac{6-2x}{2x+5}$, $\dfrac{18x+14}{21}$, $\dfrac{27-4x}{2(4x-9)}$, $\dfrac{12x-40}{33-2x}$.

2. $\dfrac{10-13x}{6}$, $x-1$, $\dfrac{20-3x}{2x-25}$, $\dfrac{14-20x}{9(x+1)}$.

3. $\dfrac{1}{x}$, 1, $\dfrac{b^2}{a^2}$, $\dfrac{a^2 + x^2}{2ax}$. 4. y, $\dfrac{4}{3x}$, $\dfrac{x(1 + x + x^3)}{1 + x^2}$.

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42. 1. $\dfrac{a(ac + b^2)}{a^2 + bc}$. 2. $\dfrac{ace}{cd - be}$. 3. $\dfrac{abc}{a+b}$. 4. $\dfrac{ace + bcd - b^2e}{b(ae - c^2)}$.

5. $\dfrac{d}{c}$. 6. $\dfrac{1}{ab}$. 7. $\dfrac{bfh}{af + 2bc - bfg}$. 8. $-\dfrac{14a}{25(a+1)}$.

9. 4. 10. $5\frac{1}{10}$.

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43. 1. 1. 2. $\frac{2}{5}$. 3. 1. 4. $-\frac{3}{5}$. 5. $\dfrac{abc}{a^2 - ab + b^2}$.

6. $1\frac{1}{3}$ 7. $\dfrac{b^2 - a^2}{4a - b}$. 8. -1. 9. $\frac{3}{5}$. 10. $\dfrac{b}{a}(a + c - b)$.

11. 1. 12. $3a$. 13. 2. 14. $-\dfrac{ab}{2a+b}$. 15. $-1\frac{4}{5}$.

16. 8. 17. 4. 18. $3\frac{3}{4}$. 19. 11. 20. 20.

21. 8. 22. 14. 23. -107. 24. $\frac{1}{4}$. 25. $-\frac{7}{8}$.

26. 0. 27. $\frac{1}{2}$. 28. $1\frac{1}{2}$. 29. 10. 30. 4.

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44. 1. 144 sq. yds. 2. 75 gals. 3. £36, £16, £8. 4. 25 of each
5. £210. 6. 22. 7. 2450, 196, 98. 8. £200.
9. 42, 66, 162. 10. 69, 81. 11. £5 8*s*. 12. 84.
13. 15 ft. by 11 ft. 14. 4*s* 8*d*. 15. 20 lb, 15 lb, 15 lb.
16. 22, £5. 17. £48. 18. $3\frac{1}{9}$ days. 19. 75. 20. £125.
21. 1504. 22. 1540, 880, 616. 23. $23\frac{1}{3}$ days.
24. $37\frac{1}{2}'$, 25′. 25. 7h. $5\frac{5}{11}'$, 6h. $16\frac{4}{11}'$. 26. 13.
27. 110 yds. 28. £72, £108. 29. $13\frac{1}{3}$ days, $2\frac{2}{3}$ days.
30. £32. 31. 10 lbs. 32. 18, $10\frac{4}{5}$, $6\frac{3}{4}$ days. 33. 40′.
34. 4*s* $4\frac{1}{2}d$, 4*s* $10\frac{1}{2}d$. 35. £48, £32, £4, £65. 36. $6\frac{2}{3}$ oz.
37. 30 hrs. 38. 40. 39. 90, 120. 40. 654.
41. 12 gals. 42. 25*s*, 20*s*. 43. 120, 104.
44. 62, 93, 155. 45. 76, 30. 46. 12, $21\frac{7}{8}s$. 47. 36′.
48. $21\frac{9}{11}$ hrs, $10\frac{10}{23}$ hrs. 49. 189. 50. $1\frac{1}{3}$ hr.

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45. 1. $x = 1, y = 1$. 2. $x = -b, y = a + b$. 3. $x = 5, y = 2$.
4. $x = \dfrac{a - b^2}{1 - ab}, y = \dfrac{b - a^2}{1 - ab}$. 5. $x = 1, y = -1$. 6. $x=1, y=0$.
7. $x=1, y=2$. 8. $x = \dfrac{bc}{a + b}, y = \dfrac{ac}{a + b}$. 9. $x=6, y=7$.
10. $x=1, y=7$. 11. $x=10, y=24$. 12. $x=144, y=216$.
13. $x=\dfrac{ac(dn+bm)}{ad+bc}, y=\dfrac{bd\,(cn-am)}{ad+bc}$. 14. $x = 2, y = 3$.
15. $x = \dfrac{bc^2}{a^2 + c^2}, y = \dfrac{a^2c}{a^2 + c^2}$. 16. $x = \dfrac{b^2 + c^2 - a^2}{2a}, y = \dfrac{a^2 + c^2 - b^2}{2b}$.
17. $x = \dfrac{ab\,(a + b)}{a^2 + b^2}, y = \dfrac{ab\,(a - b)}{a^2 + b^2}$.
18. $x = \dfrac{abc\,\{bc - a\,(b + c)\}}{b^2c^2 - a^2\,(b^2 + c^2)}, y = \dfrac{abc\,\{b\,(a + c) - ac\}}{b^2c^2 - a^2\,(b^2 + c^2)}$.
19. $x=6, y=8$. 20. $x=3, y=2$. 21. $x=5, y=2$. 22. $x=-2, y=-\frac{1}{2}$.
23. $x = 7, y = 9$. 24. $x = 5, y = 5$. 25. $x = 21, y = 20$.

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46. 1. $x = 1, y = 2, z = 3$. 2. $x = 7, y = 10, z = 9$.
3. $x = 5, y = 6, z = 7$. 4. $x = 4, y = -5, z = 6$.
5. $x = -5, y = 6, z = -2$.
6. $x = \frac{1}{2}\,(b + c - a), y = \frac{1}{2}\,(a + c - b), z = \frac{1}{2}\,(a + b - c)$.
7. $x = 1\frac{5}{7}, y = 2\frac{2}{5}, z = -12$. 8. $x = 2, y = -3, z = 4$.
9. $x = 12, y = 12, z = 12$. 10. $x = 5, y = 7, z = -3$.

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47. 1. $\frac{3}{13}$. 2. 21, 40. 3. 5*s*, 3*s*. 4. £24, £12.
5. 17 yds, 13 yds. 6. 48. 7. 108 sq ft.
8. 640, 720, 840. 9. 18*s*, 90. 10. 4 hrs, 6 hrs.
11. 75. 12. 40, 90. 13. 20, 30, 60. 14. 222.
15. 30, 50, and 70, 20: or 60, 20, and 40, 50. 16. 24, 72.
17. 72, 60. 18. 12, 12. 19. 34. 20. 31.
21. 12, 10. 22. 255. 23. 3*s*. 24. 39*s*, 21*s*, 12.

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48. 1. $x^{\frac{3}{2}} + x^{\frac{4}{3}} + x^{\frac{5}{2}} + x^{\frac{2}{3}}$; $ab^{\frac{2}{3}} + a^{\frac{2}{3}}b^{\frac{4}{3}} + a^{\frac{1}{3}}b^2 + a^{\frac{5}{3}}b$.

2. $ab^{\frac{8}{3}} + a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} + ab^{\frac{2}{3}}$; $a^{\frac{1}{2}}b^2 + ab^2 + a^{\frac{2}{3}}b^2 + a^{\frac{3}{2}}b^2$.

3. $a^{-1}+2b^{-2}+3c^{-3}+4ab^{-1}+5a^{-1}b$; $a^3b^{-3}+3a^2b^{-1}+5ab^{-2}+4a^{-2}b+2a^{-3}b^2$:

$\frac{1}{a} + \frac{2}{b^2} + \frac{3}{c^3} + \frac{4}{a^{-1}b} + \frac{5}{ab^{-1}}$; $\frac{1}{a^{-3}b^3} + \frac{3}{a^{-2}b} + \frac{5}{a^{-1}b^2} + \frac{4}{a^2b^{-1}} + \frac{2}{a^3b^{-2}}$.

4. $\frac{1}{3}a^3b^{-2}c^{-2} + 4a^{-2}b^{-1}c^2 + 2a^{-1}bc + \frac{1}{3}a^{-1}b^{-1}c^{-1}$,

$\frac{1}{2}abc^{-\frac{1}{3}} + \frac{2}{3}a^{-\frac{3}{2}}b^2c^2 + \frac{3}{4}a^{-\frac{2}{3}}b^{-\frac{1}{3}}c^{-\frac{2}{3}} + 5a^{-1}b^{-\frac{3}{4}}c$;

and $\frac{1}{3a^{-3}b^2c^2} + \frac{4}{a^2bc^{-2}} + \frac{2}{ab^{-1}c^{-1}} + \frac{1}{3abc}$,

$\frac{1}{2a^{-1}b^{-1}c^{\frac{1}{3}}} + \frac{2}{3a^{\frac{3}{2}}b^{-2}c^{-2}} + \frac{3}{4a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{2}{3}}} + \frac{5}{ab^{\frac{3}{4}}c^{-1}}$.

5. $\sqrt{a} + 2\sqrt[3]{a^2} + 3\sqrt[4]{a^3} + 4\sqrt[5]{a} + \sqrt[4]{a^3}$,

$\frac{\sqrt[4]{a}}{\sqrt[4]{b^3}} + \frac{\sqrt[3]{a^2b}}{2\sqrt{c}} + \frac{2\sqrt[4]{ac^3}}{3\sqrt{b^3}} + \frac{\sqrt[5]{b^2c^3}}{4\sqrt[5]{a}} + \frac{\sqrt[6]{bc^5}}{5\sqrt[4]{a^3}}$.

6. $\frac{bc}{a} + \frac{ac}{b^2} + \frac{1}{abc} + \frac{c^3}{ab^2}$, $\frac{1}{\sqrt[5]{a^2}} + \frac{\sqrt{a}}{\sqrt[3]{b^4}} + \frac{\sqrt[3]{b^2}}{\sqrt{a^3}} + \frac{1}{\sqrt[3]{b^5}}$.

7. $\frac{c}{a^2b^2} + 2abc + \frac{3a^3}{bc^2} + ab^2c^3$; $\frac{\sqrt[3]{b}}{a^2} + \frac{\sqrt{a^3}}{\sqrt[3]{b^2}} + \frac{\sqrt[3]{a^2}}{\sqrt[4]{b^3}} + \frac{b^2}{\sqrt{a}}$.

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49. 1. $ab^{-\frac{2}{3}}$, $a^{\frac{1}{2}}$, $a^{-\frac{1}{4}}$, a^2b. 2. $x^{-4}y^4$, $x^2y^{\frac{1}{4}}$. 3. x^3y^2z, a^2b^6c.

4. $x^2+6xz^{\frac{1}{3}}+9z^{\frac{2}{3}}-4y$. 5. $a-b^2$. 6. a^3-64b^2.

7. $a^3-a^{\frac{4}{3}}+2a^{\frac{1}{3}}-2-a^{-\frac{2}{3}}+a^{-3}$. 8. $x+x^{\frac{3}{4}}y^{-\frac{1}{4}}-x^{\frac{1}{4}}y^{-\frac{3}{4}}-y^{-1}$.

9. $8x^{\frac{3}{4}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} + 2x^{\frac{1}{4}}y + y^{\frac{3}{2}}$; $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

10. $a^{-\frac{5}{2}} - 2a^{-2}b^{\frac{1}{3}} + 4a^{-\frac{3}{2}}b^{\frac{2}{3}} - 8a^{-1}b + 16a^{-\frac{1}{2}}b^{\frac{4}{3}} - 32b^{\frac{5}{3}}$,

$x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$.

11. $4a - 2a^{\frac{1}{2}}b^{-\frac{1}{2}} + 2a^{\frac{1}{2}}c^{\frac{1}{3}} + b^{-1} + b^{-\frac{1}{2}}c^{\frac{1}{3}} + c^{\frac{2}{3}}$.

12. $ab^{-3} + 3a^{\frac{1}{3}}b^{-1} + 3a^{-\frac{1}{3}}b + a^{-1}b^3$, $\frac{8}{27}x^2y^{-1} - 2x + \frac{9}{2}y - \frac{27}{8}x^{-1}y^2$.

13. $a^2 - 6a^{\frac{5}{3}}b^{\frac{1}{6}} + 21a^{\frac{4}{3}}b^{\frac{1}{3}} - 44ab^{\frac{1}{2}} + 63a^{\frac{2}{3}}b^{\frac{2}{3}} - 54a^{\frac{1}{3}}b^{\frac{5}{6}} + 27b$.

14. $a^{\frac{4}{3}} - 4a + 10a^{\frac{2}{3}} - 16a^{\frac{1}{3}} + 19 - 16a^{-\frac{1}{3}} + 10a^{-\frac{2}{3}} - 4a^{-1} + a^{-\frac{4}{3}}$.

15. $x^3 - 4x^{\frac{9}{4}}y^{\frac{5}{2}} + 6x^{\frac{3}{2}}y^5 + 4x^{\frac{3}{4}}y^{\frac{15}{2}} + y^{10}$,
$x^{\frac{15}{4}} - 5x^3y^{\frac{5}{2}} + 10x^{\frac{9}{4}}y^5 - 10x^{\frac{3}{2}}y^{\frac{15}{2}} + 5x^{\frac{3}{4}}y^{10} - y^{\frac{25}{2}}$;
$a^2b^{-2} - 4ab^{-1} + 6 - 4a^{-1}b + a^{-2}b^2$,
$a^{\frac{5}{2}}b^{-\frac{5}{2}} - 5a^{\frac{3}{2}}b^{-\frac{3}{2}} + 10a^{\frac{1}{2}}b^{-\frac{1}{2}} - 10a^{-\frac{1}{2}}b^{\frac{1}{2}} + 5a^{-\frac{3}{2}}b^{\frac{3}{2}} - a^{-\frac{5}{2}}b^{\frac{5}{2}}$.

16. $ab^{-1} + 1 + a^{-1}b$. 17. $a^{\frac{2}{3}} - \frac{3}{2}a^{\frac{1}{3}} + 3 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}}$.

18. $a^{-\frac{1}{2}}x^{\frac{1}{2}} - 1 + a^{\frac{1}{2}}x^{-\frac{1}{2}}$. 19. $xy^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{2}{3}}$. 20. $2x^{\frac{3}{2}} - 3y^{\frac{3}{4}}$.

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50. 1. $64^{\frac{1}{4}}$, $81^{\frac{1}{5}}$, $(\frac{1}{9})^{\frac{1}{3}}$, $(\frac{1}{8})^{\frac{1}{4}}$, $(\frac{3}{2})^{\frac{1}{4}}$, $8^{\frac{1}{5}}$.

2. $25^{\frac{1}{2}}$, $(\frac{25}{4})^{\frac{1}{2}}$, $(\frac{4}{9}a^2)^{\frac{1}{2}}$, $(\frac{9}{4}a^4)^{\frac{1}{2}}$, $\{\frac{1}{4}(a^2 + 2ab + b^2)\}^{\frac{1}{2}}$;
$125^{\frac{1}{3}}$, $(\frac{125}{8})^{\frac{1}{3}}$, $(\frac{8}{27}a^3)^{\frac{1}{3}}$, $(\frac{27}{8}a^6)^{\frac{1}{3}}$, $\{\frac{1}{8}(a^3 + 3a^2b + 3ab^2 + b^2)\}^{\frac{1}{3}}$.

3. $(\frac{1}{729})^{\frac{1}{3}}$, $(\frac{27}{1000})^{\frac{1}{3}}$, $\left(\frac{1}{a^6}\right)^{\frac{1}{3}}$, $\left(\frac{a^3}{b^3c^6}\right)^{\frac{1}{3}}$;
$6561^{-\frac{1}{4}}$, $(\frac{10000}{81})^{-\frac{1}{4}}$, $(a^8)^{-\frac{1}{4}}$, $\left(\frac{b^4c^8}{a^4}\right)^{-\frac{1}{4}}$.

4. $\sqrt{125}$, $\sqrt{3}$, $\sqrt{12}$, $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{320}$.

5. $\sqrt[3]{54}$, $\sqrt[3]{256}$, $\sqrt[4]{2048}$, $\sqrt[4]{3}$, $\sqrt[3]{\frac{2}{3}}$, $\sqrt[6]{\frac{1}{18}}$.

6. $\sqrt{4a}$, $\sqrt{98a^2x}$, $\frac{1}{b}$, $\sqrt{\frac{a+b}{a-b}}$, $\frac{1}{a^2+ab+b^2}$.

7. $\sqrt{2ab}$, $\sqrt{6a^3x}$, $\sqrt[3]{\frac{4a^2}{9b^2}}$, $\sqrt[3]{\frac{2a}{3}}$, $\sqrt{a^2 - x^2}$.

8. $3\sqrt{5}$, $5\sqrt{5}$, $36\sqrt{3}$, $3\sqrt[3]{5}$, $18\sqrt[3]{2}$, $\frac{1}{2}\sqrt{6}$, $\sqrt[3]{12}$, $\sqrt[4]{54}$, 6.

9. $4\sqrt[4]{2}$, $8\sqrt[3]{2}$, $6\sqrt[5]{48}$, $\frac{2}{3}\sqrt{2}$, $\frac{2}{27}\sqrt{2}$, $\frac{2}{9}\sqrt{2}$, $\frac{3}{7}\sqrt{21}$, $\frac{3}{2}\sqrt[3]{150}$, $\sqrt[4]{375}$.

10. $2\sqrt{3}$, $15\sqrt{3}$, $\frac{7}{2}\sqrt{3}$, $\frac{4}{45}\sqrt{3}$, $\frac{1}{2}\sqrt{3}$, $\frac{1}{6}\sqrt{3}$.

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51. 1. $\sqrt{108}$, $\sqrt{112}$; $\sqrt[3]{81}$, $\sqrt[3]{80}$; $\sqrt[3]{120}$, $\sqrt[3]{128}$, $\sqrt[3]{135}$; $\sqrt[6]{125}$, $\sqrt[2]{121}$;
$\sqrt[4]{\frac{1}{4}}$, $\sqrt[4]{\frac{1}{3}}$; $\sqrt[6]{125}$, $\sqrt[6]{144}$, $\sqrt[6]{162}$.

2. $\sqrt{2}$, $3\sqrt{5}$. 3. $\frac{29}{2}\sqrt{3}$, $9\sqrt[3]{9}$. 4. $24\sqrt{3}$, $120\sqrt{3}$, 36.

5. $216\sqrt[12]{6}$, $288\sqrt[12]{72}$. 6. $5 - \sqrt{6}$, $6\sqrt{3} + 3\sqrt{30}$.

7. 16. 8. $\frac{1}{3}(\sqrt{2}+\sqrt{3}+\sqrt{5})$, $\frac{1}{3}\sqrt{6}+\frac{1}{2}\sqrt[6]{32}+\frac{1}{6}\sqrt[4]{120}$.
9. $\frac{1}{5}(2\sqrt{2}+\sqrt{3})$, $\sqrt{5}+1$, $\sqrt{5}-\sqrt{2}$, $4+\sqrt{2}$, $\frac{1}{2}(7+3\sqrt{5})$, $\frac{1}{11}(7\sqrt{14}-13)$.
10. $\frac{1}{27}(58+8\sqrt{7})$, $\frac{1}{19}(8\sqrt{5}+23)$, $\frac{1}{6}(3-\sqrt{6})$.
11. $\dfrac{a+\sqrt{a^2-x^2}}{x}$, $\dfrac{2\sqrt{a^2-x^2}}{x^2}$, $4x\sqrt{x^2-1}$. 12. $2x^2$, $\dfrac{1}{1-x^2}$.

52. 1. $\sqrt{3}+1$. 2. $3+\sqrt{2}$. 3. $\sqrt{5}-\sqrt{3}$. 4. $2\sqrt{5}-3\sqrt{2}$.
5. $4\sqrt{2}-3$. 6. $\frac{1}{2}\sqrt{5}-1$. 7. $2-\frac{1}{3}\sqrt{3}$. 8. $\frac{2}{3}\sqrt{2}-\frac{1}{4}$.
9. $\sqrt{2}+1$. 10. $\sqrt{5}-1$. 11. $\frac{1}{2}(\sqrt{5}+1)$. 12. $\sqrt{5}+\frac{1}{2}\sqrt{3}$.

53. 1. $\frac{9}{20}$. 2. $\dfrac{\sqrt{a}}{2+\sqrt{a}}$. 3. $\dfrac{ab}{1-2\sqrt{b}}$. 4. $\dfrac{1}{b-2}$. 5. 2.
6. $\dfrac{2ab-b^2}{2(a-b)}$. 7. a^2-b. 8. $\frac{9}{16}a$. 9. $\dfrac{(a-b)^2}{2b}$. 10. $\dfrac{b(b-2a)}{3b-2a}$.

54. 1. ± 2. 2. ± 3. 3. ± 1. 4. $\pm\frac{1}{2}$. 5. $\pm\frac{1}{2}$. 6. $\pm 2\frac{1}{3}$.
7. $\pm\frac{2}{3}$. 8. ± 5. 9. ± 3. 10. ± 5. 11. ± 2. 12. ± 2.
13. $\pm\sqrt{3}$. 14. $\pm\dfrac{a\sqrt{3}}{2}$. 15. $\pm\sqrt{\dfrac{a^2(c-d)^2-b^2(c+d)^2}{2(c^2+d^2)}}$. 16. $\dfrac{(n-1)a}{\sqrt{2n-1}}$.

55. 1. 4, −2. 2. −1, −9. 3. 20, −6.
4. 7, 5. 5. 8, −40. 6. 10, −110.

56. 1. 1, −8. 2. 17, −4. 3. −5, −20.
4. −1, −12. 5. 1, −26. 6. 25, −136.

57. 1. 6, $-5\frac{2}{3}$. 2. 6, $-4\frac{1}{2}$. 3. $8\frac{3}{5}$, −10.
4. 14, $-10\frac{2}{7}$. 5. 12, $-12\frac{1}{12}$. 6. 13, $-11\frac{4}{13}$.

58. 1. 10, 2. 2. 3, −1. 3. 2, $-\frac{3}{7}$. 4. $1\frac{1}{2}$, $-\frac{15}{22}$.
5. $1\frac{2}{3}$, $-1\frac{1}{2}$. 6. 7, $-1\frac{1}{2}$. 7. 2, $\frac{1}{2}$. 8. $\frac{1}{2}(-9\pm 3\sqrt{3})$.
9. 2, $\frac{12}{25}$. 10. 3, $-\frac{4}{5}$. 11. $\frac{1}{8}(27\pm\sqrt{57})$. 12. 2, −3.

59. 1. 11, −13. 2. 5, $5\frac{2}{5}$. 3. 5, 21. 4. 7, $-1\frac{3}{7}$.
5. 6, $3\frac{1}{13}$. 6. 5, $-4\frac{3}{4}$. 7. 1, $10\frac{2}{3}$. 8. 3, $-8\frac{7}{10}$.

60. 1. $x^2-4x-21=0$. 2. $6x^2+5x-6=0$.
3. $16x^4-153x^2+81=0$. 4. $x^4-6x^3+11x^2-6x=0$.
5. $4x^4-7x^3-3x=0$. 6. $4x^5+3x^4-17x^3-12z^2+4x=0$.

61. 1. $x=7,\ y=\pm 4$. 2. $x=4,\ y=-3,\ x=-3,\ y=4$. 3. $x=4,\ y=3,\ x=+1\frac{10}{23},\ y=\frac{17}{23}$.

4. $x=8,\ y=2\frac{1}{2},\ x=-2\frac{1}{2},\ y=-8$. 5. $x=6,\ y=5,\ x=-6,\ y=-5$. 6. $x=5,\ y=3,\ x=\frac{3}{4},\ y=-1\frac{1}{4}$.

7. $x=5,\ y=3,\ x=1\frac{7}{10},\ y=-\frac{3}{10}$. 8. $x=3,\ y=4,\ x=-1\frac{1}{11},\ y=-2\frac{4}{11}$.

9. $x=4,\ y=2,\ x=2,\ y=4$. 10. $x=10,\ y=15,\ x=-10\frac{7}{9},\ y=-16\frac{1}{6}$.

11. $x=3,\ y=2,\ x=-2,\ y=-3$. 12. $x=5,\ y=4,\ x=4,\ y=5$.

13. $x=\frac{1}{2}\{a\pm\sqrt{2b^2-a^2}\},\ y=\frac{1}{2}\{a\pm\sqrt{2b^2-a^2}\}$. 14. $x=\frac{1}{2}\{\pm\sqrt{4a^2+b^2}+b\},\ y=\frac{1}{2}\{\pm\sqrt{4a^2+b^2}-b\}$.

15. $x=8,\ y=1,\ x=1,\ y=8$. 16. $x=\pm\dfrac{a^2}{\sqrt{a^2+b^2}},\ y=\pm\dfrac{b^2}{\sqrt{a^2+b^2}}$.

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62. 1. ± 12, ± 15. 2. ± 10, ± 16. 3. ± 4, ± 12.
4. 15 yds, 25 yds. 5. 8 and 6, or 56 and − 42.
6. 27 yds. 7. 4550. 8. 24 or − 3.
9. 4 or $-1\frac{1}{3}$. 10. 40 yds by 24. 11. 9, 12, 15.
12. 12 and 7, or $-9\frac{1}{2}$ and $-14\frac{1}{2}$. 13. 10 yds, 16 yds. 14. 3 in.
15. 26 ft, 38 ft. 16. 16. 17. 49, £3. 18. 4 ft, 5 ft.
19. £60 or £40. 20. 10, 15. 21. £275, £225.
22. 25, 20. 23. 264. 24. 2, 5, 8.

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63. 1. $x=3,\ y=1$; $x=23,\ y=2$; $x=31, 2,\ y=2, 5$; $x=30, 15,\ y=1, 8$.
2. $x=5,\ y=6$; $x=5,\ y=12$; $x=49,\ y=22$; $x=11,\ y=3$. 3. 3, 5, 5. 9.
4. $x=5,\ y=3,\ z=6$. 5. 5. 6. 4, 2. 7. 4.
8. *A* gives 14 pieces, *B* 9. 9. 8; 16. 10. 21, 12. 11. 4.
12. 59. 13. 8 h. g. and 3 h. c. 14. £13 1*s* or £42 1*s*.
15. By paying £5 and receiving 4 *louis*.
16. 3, 21, 16, or 6, 2, 32. 17. 503.
18. 2*s*, 4*s*, 5*s*. 19. 209. 20. 301.

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64. 1. 32, 272. 2. 39, 400. 3. 63, 363. 4. 694, 34750.
5. 9. 16. 6. − 1, 0. 7. − 28. 8. − 275.
9. $16\frac{1}{4}$. 10. $-84\frac{1}{2}$. 11. $336\frac{2}{3}$. 12. − 84.

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65. 1. 12. 2. 5. 3. 20. 4. $-\frac{8}{77}$.

5. 5, 8, 11, 14; −2, −6, −10, −14.

6. $3\frac{3}{5}, 4\frac{1}{5}, 4\frac{4}{5}, 5\frac{2}{5}, 6, 6\frac{3}{5}, 7\frac{1}{5}, 7\frac{4}{5}, 8\frac{2}{5}$; −11, −9, −7, −5, −3, −1, +1.

7. 4, 15, 26, 37, 48, 59, 70, 81, 92, 103; $-2\frac{3}{4}, -2\frac{1}{2}, -2\frac{1}{4}, -2, -1\frac{3}{4}, -1\frac{1}{2}, -1\frac{1}{4}, -1$.

8. $-2, -1\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}, 1, 1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}, 4$; $-2\frac{4}{5}, -2\frac{1}{5}, -1\frac{3}{5}, 1, -\frac{2}{5}, \frac{1}{5}, \frac{4}{5}, 1\frac{3}{5}, 2$. 9. 5, 7, 9. 10. $-3\frac{1}{3}, 3\frac{1}{3}$, 10.

11. $-\frac{3}{4}, \frac{1}{4}, 1\frac{1}{4}$. 12. n^2. 13. 300.

14. 78, 90. 15. £5 3*s*; £135 4*s*. 16. 5 miles, 1300 yards.

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66. 1. 64, 85. 2. 1280, 1705. 3. 96, 189. 4. −256, −170.

5. 4096, 3277. 6. −512, −341. 7. $\frac{35}{128}$. 8. $1\frac{179}{486}$.

9. $4\frac{17}{162}$. 10. $2\frac{247}{432}$. 11. $5\frac{394}{729}$. 12. $72\frac{4}{25}$.

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67. 1. 8. 2. $1\frac{1}{2}$. 3. $\frac{1}{5}$. 4. $\frac{9}{10}$. 5. $\frac{2}{3}$. 6. $\frac{5}{7}$.

7. $\frac{3}{10}$. 8. 1. 9. $1\frac{5}{11}$. 10. $1\frac{1}{5}$. 11. $10\frac{1}{8}$. 12. $-2\frac{2}{15}$.

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68. 1. 4. 2. 3, 6, 12, &c.

3. 4, −8, 16, &c.; or $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}$, &c. 4. $\frac{1}{10}$.

5. 3, 15, 75, 375; or −2, 10, −50, 250.

6. ±4, 8, ±16; ±2, 8, ±32.

7. $\frac{1}{5}, -\frac{3}{5}, \frac{4}{5}, -1\frac{3}{5}$; $-1, 1\frac{1}{2}, -2\frac{1}{4}, 3\frac{3}{8}$.

8. $2 + \frac{2}{3} + \frac{3}{9}$ + &c.; or $4 - \frac{4}{3} + \frac{4}{9}$ − &c.

9. $3 - \frac{3}{2} + \frac{3}{4}$ − &c. 10. $1\frac{7}{20}$ or $6\frac{3}{4}$.

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69. 1. $-4, \infty, 4, \ldots \frac{4}{5}, \frac{2}{3}, \frac{4}{7}$; $\frac{15}{19}, \frac{15}{16}, 1\frac{2}{13}, \ldots 15, -7\frac{1}{2}, -3$; $\frac{7}{10}, \frac{7}{9}, \frac{7}{8}, \ldots 1\frac{3}{4}, 2\frac{1}{3}, 3\frac{1}{2}$.

2. $2\frac{2}{5}$, 3; $1\frac{1}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$. 3. 24.

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70. 1. $3\frac{1}{4}, 3, 2\frac{10}{13}$. 2. $2\frac{7}{16}, 2\frac{1}{4}, 2\frac{1}{13}$. 3. 8 and 2.

4. 1 or 16. 5. 8 and 2. 6. 9 and 1, or $\frac{4}{5}$ and $-7\frac{1}{5}$.

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71. 1. $\frac{15}{20}, \frac{16}{20}, \frac{156}{168}, \frac{161}{168}, \frac{405}{1155}, \frac{735}{1155}, \frac{847}{1155}$. 2. $\frac{a+b}{a-b}$.

3. $\frac{4x}{x+y}, \frac{x^3+y^3}{x^2+y^2}, x^2+y^2$. 4. $\frac{4}{15}, \frac{2}{3}$. 5. $\frac{a^4+a^2x^2+x^4}{a^4-x^4}$.

6. $\frac{x^2-11x+28}{x^2}$. 7. 1. 8. $\frac{(a+b)^3}{b^2(a-b)}$. 9. $\frac{ad-bc}{c-d}$.

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72. 1. 10, $4\frac{1}{6}$, $2\frac{3}{16}$. 2. 9, $4\frac{1}{2}$, $\frac{15}{49}$. 3. 6, $1\frac{3}{5}$, $1\frac{2}{3}$.

13. (i) $x = b\left(\frac{a+b}{a-b}\right)^2$; (ii) $x = a + b$ or $\frac{1}{2}(a-b)$; (iii) $x=1$, $y=4$; (iv) $x = \pm 9$, $y = \pm 3$. 14. 3. 15. 25, 20.

16. 8 : 7. 17. £200, £150. 18. 300.

19. £125, £$166\frac{2}{3}$, £$208\frac{1}{3}$; £$212\frac{36}{47}$, £$159\frac{27}{47}$, £$127\frac{31}{47}$. 20. 6.

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73. 1. $xy = \frac{12}{25}(x^2+y^2)$. 2. 2. 3. $y = \frac{14}{4-5x}$. 4. $y = 3x + 2x^2 + x^3$.

5. $y = x^2 + 2x + 3$. 6. $z = \frac{3}{10}x + \frac{2}{5}x^2$. 8. $\frac{1}{2}AC.BC$.

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74. 1. 720, 720. 2. 5040. 3. 6720, 45360, 3326400, 19958400.

4. 12600. 5. 9. 6. 1120, 831600, 336, 34650.

7. 6. 8. 7. 9. 15. 10. 3628800. 11. 6. 12. 4.

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75. 1. 126, 84, 36. 2. 330, 330, 11. 3. 3003, 455. 4. 6.

5. 63. 6. 210, 84. 7. 50063860, 5006386. 8. 18. 9. 12.

10. 11. 11. 12. 12. 43092000.

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76. 1. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

2. $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$.

3. $1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8$.

4. $a^9 - 9a^8x + 36a^7x^2 - 84a^6x^3 + 126a^5x^4 - 126a^4x^5 + 84a^3x^6 - 36a^2x^7 + 9ax^8 - x^9$.

5. $1 + 12x + 66x^2 + 220x^3 + 495x^4 + 729x^5 + 924x^6 + 792x^7 + 495x^8 + 220x^9 + 66x^{10} + 12x^{11} + x^{12}$.

6. $1 - 20x + 180x^2 - 960x^3 + 3360x^4 - 8064x^5 + 13440x^6 - 15360x^7 + 11520x^8 - 5120x^9 + 1024x^{10}$.

7. $a^6 - 18a^5x + 135a^4x^2 - 540a^3x^3 + 1215a^2x^4 - 1458ax^5 + 729x^6$.

8. $256x^8 + 1024ax^7 + 1792a^2x^6 + 1792a^3x^5 + 1120a^4x^4 + 448a^5x^3 + 112a^6x^2 + 16a^7x + a^8$.

9. $128a^7 - 1344a^6x + 6048a^5x^2 - 15120a^4x^3 + 22680a^3x^4 - 20412a^2x^5 + 10206ax^6 - 2187x^7$.

10. $1 - 5x + \frac{45}{4}x^2 - 15x^3 + \frac{105}{8}x^4 - \frac{63}{8}x^5 + \frac{105}{32}x^6 - \frac{15}{16}x^7 + \frac{45}{256}x^8 - \frac{5}{256}x^9 + \frac{1}{1024}x^{10}$.

11. $1 - \frac{11}{3}x + \frac{55}{9}x^2 - \frac{55}{9}x^3 + \frac{110}{27}x^4 - \frac{154}{81}x^5 + \frac{154}{243}x^6 - \frac{110}{729}x^7 + \frac{55}{2187}x^8 - \frac{55}{19683}x^9 + \frac{11}{59049}x^{10} - \frac{1}{177147}x^{11}$.

12. $\frac{1}{4096}x^{12} - \frac{1}{512}x^{11}y + \frac{11}{1536}x^{10}y^2 - \frac{55}{3456}x^9y^3 + \frac{55}{2304}x^8y^4 - \frac{11}{432}x^7y^5 + \frac{77}{3888}x^6y^6 - \frac{11}{972}x^5y^7 + \frac{55}{11664}x^4y^8 - \frac{55}{39366}x^3y^9 + \frac{11}{39366}x^2y^{10} - \frac{2}{59049}xy^{11} + \frac{1}{531441}y^{12}$.

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77. 1. $1 - x + x^2 - x^3 + x^4 -$ &c. 2. $1 + 3x + 9x^2 + 27x^3 + 81x^4 +$ &c.
3. $1 - 6x + 27x^2 - 108x^3 + 405x^4 -$ &c.
4. $1 + 6x + 24x^2 + 80x^3 + 240x^4 +$ &c.
5. $1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \frac{5}{16}x^4 +$ &c. 6. $1 - x + \frac{2}{3}x^2 - \frac{10}{27}x^3 + \frac{5}{27}x^4 -$ &c.
7. $1 - x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 +$ &c. 8. $1 - 2x - x^2 - \frac{4}{3}x^3 - \frac{7}{3}x^4 -$ &c.
9. $1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{35}{16}x^3 + \frac{315}{128}x^4 +$ &c.
10. $1 - \frac{7}{3}x^2 + \frac{14}{9}x^4 - \frac{14}{81}x^6 - \frac{7}{243}x^6 -$ &c.
11. $1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \frac{35}{243}x^4 +$ &c.
12. $1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8 -$ &c.

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78. 1. $\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \frac{5}{64}x^4 +$ &c.
2. $\frac{1}{27} + \frac{2}{27}x + \frac{8}{81}x^2 + \frac{80}{729}x^3 + \frac{80}{729}x^4 +$ &c.
3. $a^{-1} - a^{-2}bx + a^{-3}b^2x^2 - a^{-4}b^3x^3 + a^{-5}b^4x^4 -$ &c.
4. $a^{-2} + 2a^{-3}b^2x + 3a^{-4}b^4x^2 + 4a^{-5}b^6x^3 + 5a^{-6}b^8x^4 -$ &c.
5. $a^2 + 6a^{\frac{7}{3}}b^{\frac{1}{3}} + 21a^{\frac{8}{3}}b^{\frac{2}{3}} + 56a^3b + 126a^{\frac{10}{3}}b^{\frac{4}{3}} +$ &c.
6. $a^{\frac{2}{5}} - \frac{1}{5}a^{-\frac{8}{5}}x^2 - \frac{2}{25}a^{-\frac{18}{5}}x^4 - \frac{6}{125}a^{-\frac{28}{5}}x^6 - \frac{21}{625}a^{-\frac{38}{5}}x^8 -$ &c.
7. $a - 3a^{\frac{4}{3}}b^{\frac{1}{3}} + 6a^{\frac{5}{3}}b^{-\frac{2}{3}} - 10a^2b^{-1} + 15a^{\frac{7}{3}}b^{-\frac{4}{3}} -$ &c.
8. $a^{\frac{1}{3}} - \frac{1}{3}a^{-\frac{2}{3}}x - \frac{1}{9}a^{-\frac{5}{3}}x^2 - \frac{5}{81}a^{-\frac{8}{3}}x^3 - \frac{10}{243}a^{-\frac{11}{3}}x^4 -$ &c.
9. $a^{-1} + \frac{1}{5}a^{-6}x^5 + \frac{3}{25}a^{-11}x^{10} + \frac{11}{125}a^{-16}x^{15} + \frac{44}{625}a^{-21}x^{20} +$ &c.
10. $a^{\frac{8}{3}} - \frac{4}{3}a^{\frac{2}{3}}x^2 + \frac{2}{9}a^{-\frac{4}{3}}x^4 + \frac{4}{81}a^{-\frac{10}{3}}x^6 + \frac{5}{243}a^{-\frac{16}{3}}x^8 +$ &c.
11. $a^{-\frac{3}{2}} + \frac{3}{4}a^{-\frac{7}{2}}x^2 + \frac{21}{32}a^{-\frac{11}{2}}x^4 + \frac{77}{128}a^{-\frac{15}{2}}x^6 - \frac{1155}{2048}a^{-\frac{19}{2}}x^8 +$ &c.
12. $a^{-\frac{1}{3}}x^{-\frac{1}{3}} + \frac{1}{3}a^{-\frac{4}{3}}x^{\frac{2}{3}} + \frac{2}{9}a^{-\frac{7}{3}}x^{\frac{5}{3}} + \frac{14}{81}a^{-\frac{10}{3}}x^{\frac{8}{3}} + \frac{35}{243}a^{-\frac{13}{3}}x^{\frac{11}{3}} +$ &c.

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79. 1. 5221, 203116. 2. 100101100, 102010, 10230, 2200, 1220.
3. 41104, 23420, 14641, 7571, 5954. 4. 235, 1465.
5. 511, 22154. 6. 1212, 1212201.

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80. 1. 1 + 14 + 244 + 4344 + 114144 + 2050544 = 2214223 (sen.) = 111111 (den.).
2. 100001000000 (bin.) = 201000 (quat.).
3. 1756 × 345 = 701746 (oct.), 1337 × 274 = 381011 (non.), 345, 274. 4. 57264, 95494, *e7t8*.
5. 4112, 6543, 62*te*. 6. 1295, 216; 2400, 343; 4095, 512.

MISCELLANEOUS EXAMPLES: PART I.

1. $(a^2 - b^2)b^2 + (a^3 - 3ab^2 + b^3)x - (2a - b)ax^2$.

2. $3x^2 - 2abx - 2a^2b^2$. 3. $3\frac{1}{7}$. 4. $(m + n)a$, $\dfrac{x^2 - xy + y^2}{x^2 + xy + y^2}$.

5. $1\frac{2}{13}$, 8.152. 6. $\dfrac{13 - 9x}{11 - 8x}$, $\dfrac{x + 3}{x^4 - 1}$. 7. 98, $\frac{1}{4}n(3n + 25)$.

8. $1\frac{7}{17}$. 9. $5\sqrt{5}$, $7a\sqrt{2x}$, $\sqrt[3]{4}$. 10. $1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\frac{35}{8}x^4+$&c.

11. (i) $x = 5$; (ii) $x = 5$ or $-1\frac{1}{2}$; (iii) $x = 4$, $y = 3$;
(iv) $x = \pm 3$, $y = \pm 2$, or $x = \pm 2$, $g = \pm 3$. 12. $\frac{7}{8}$.

13. $1+4x+2x^2-5x^3-x^4+x^5+\frac{1}{4}x^6$. 14. $-2x^2+8xy-5y^2$. 15. 68.

16. $x^2 - y^2$. 17. 125, 1.709. 18. x, $\dfrac{a + x}{x - y}$. 19. $2\frac{7}{27}$, $2\frac{1}{4}$.

20. $a^{\frac{3}{2}}b^{-2}c^{\frac{9}{4}}$, $x^{-\frac{1}{2}}z^{\frac{2}{3}}$. 21. $a^{-\frac{1}{3}}\{1+a^{-1}x+2a^{-2}x^2+\frac{14}{3}a^{-3}x^3+\frac{35}{3}a^{-4}x^4+\text{\&c.}\}$.

22. 1232, 11313, 363, 1044.

23. (i) $x = 3\frac{3}{16}$; (ii) $x = 2$ or $-\frac{2}{3}$; (iii) $x = 5$, $y = 4\frac{1}{3}$.

24. 12 days. 25. $x^6 - a^3$. 26. $\frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$. 27. $\frac{3}{4}$.

28. $\dfrac{x^2}{x^2 - y}$, $\dfrac{3a + 1}{4a^2 + 2a - 1}$. 29. $\dfrac{4 - x}{5 - x}$, $\dfrac{2x}{x^4 - 1}$.

30. 139, 1.39, 4.3955. 31. $2\frac{3}{16}$, .051. 32. $\frac{24}{55}\{1 - (-\frac{5}{6})^n\}$, $\frac{24}{55}$.

33. $(ax)^{-\frac{3}{5}}\{1 + \frac{3}{5}a^{-1}x + \frac{12}{25}a^{-2}x^2 + \frac{52}{125}a^{-3}x^3 + \frac{234}{625}a^{-4}x^4 + \text{\&c.}\}$.

34. 7; 22 dollars and 57 doubloons.

35. (i) $x = 17$; (ii) $x = 60$, $y = 40$; (iii) $x = 3$ or $-\frac{40}{59}$.

36. $3\frac{3}{4}$ hrs. 37. $2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$.

38. $1 - \frac{1}{6}x + \frac{7}{36}x^2 + \frac{5}{216}x^3 + \frac{73}{1296}x^4 +$ &c. 39. $\frac{1}{8}$.

40. $\dfrac{x^2 - x + 1}{x^2 + x + 1}$. 41. 139, .6933. 42. $\dfrac{3x}{1 - x}$.

43. $a^7\{1 - 7a^{-2}x^2 + \frac{21}{2}a^{-4}x^4 + \frac{7}{2}a^{-6}x^6 + \frac{35}{8}a^{-8}x^8 + \text{\&c.}\}$.

44. $\frac{1}{3}\sqrt[3]{18a}$, $\sqrt[4]{5x^2y^2}$. 45. 93, $\frac{1}{12}(n^3 + n - 6)$.

46. 5221, 40255141, 6252711, 2451, 3341584, 1828.

47. (i) $x = 2\frac{2}{3}$; (ii) $x = 39$, $y = 21$, $z = 12$;
(iii) $x = a.\dfrac{a + b}{a - b}$, $y = a.\dfrac{a - b}{a + b}$. 48. 64 days. 49. $a - b$.

50. $9 + 1 + 49 = 59$. 51. $3(a^2 + b^2 + c^2) - 2(ab + ac + bc)$.

52. $x^2 - 9y^2$. 53. 1054, $\sqrt{7} + \sqrt{2}$.

54. $\frac{b}{a}$, $\frac{a-b}{a+b}$. 55. $\frac{2}{9}\{1-(\frac{1}{10})^n\}$, $\frac{2}{9}$. 56. 6.

57. $a^{-\frac{4}{5}}\{1+\frac{2}{5}a^{-2}bx+\frac{7}{25}a^{-4}b^2x^2+\frac{28}{125}a^{-6}b^3x^3+\frac{119}{625}a^{-8}b^4x^4+\&c.\}$ 58. 21.

59. (i) $x=-\frac{3}{5}$; (ii) $x=3$, $y=4$; (iii) $x=3$, $y=1$; (iv) $x=-1$, $y=-3$.

60. 16. 61. $26x^{\frac{1}{2}}y^{\frac{1}{2}}-x^{\frac{1}{4}}y-7x^{\frac{3}{4}}-10y^{\frac{3}{2}}$. 62. x^2+2x+1.

63. 1. 64. $a^2b^2(b+c)$. 65. $2x-1$. 66. $\frac{1-x}{x(4x^2-1)}$.

67. 12.747. 68. $(a^2x)^{-\frac{1}{2}}\{1+\frac{1}{2}a^{-1}x+\frac{3}{8}a^{-2}x^2+\frac{5}{16}a^{-3}x^3+\frac{35}{128}a^{-4}x^4+\&c.\}$

69. 30. 70. 250, 60300, 13874000.

71. (i) $x=9$; (ii) $x=3\frac{1}{2}$ or -4; (iii) $x=\frac{1}{2}$, $y=\frac{1}{3}$.

72. 8 hrs; $17\frac{1}{7}$ hrs, 24 hrs, 40 hrs. 73. $6(x+2x^{\frac{1}{3}}+4x^{-\frac{1}{3}}+8x^{-1})$.

74. $\frac{1}{8}$. 75. x, $\frac{x^3+1}{x^2-3x+1}$. 76. $55\sqrt{7}$. 77. 1.772452, $x^{-\frac{1}{2}}$.

79. $50\frac{2}{5}$, $\frac{1}{5}n(3n+1)$. 80. $x^6-2x^5-8x^3+8x+16=0$. 81. 28.

82. 15120, 120. 83. (i) $x=1$; (ii) $x=2\frac{2}{3}$, $y=5\frac{1}{3}$; (iii) $x=3$.

84. £$553\frac{1}{3}$, £$1106\frac{2}{3}$, £3320. 85. x^3+1+x^{-3}. 86. $a^2+ax-2x^2$.

87. 7. 88. $\frac{1}{2}(1+x)$, $\frac{3a+2b}{3b-2a}$. 89. $a^{\frac{2}{3}}x^{-\frac{2}{3}}-a^{\frac{1}{3}}x^{-\frac{1}{3}}b^{\frac{3}{2}}+b^{\frac{4}{3}}$.

90. $x^{\frac{1}{2}}-1-x^{-\frac{1}{2}}$. 91. $\frac{1}{5}n(3n+1)$, $\frac{8}{15}\{\frac{5^n}{2}-1\}$.

92. $1+x-\frac{3}{2}x^2+\frac{7}{2}x^3-\frac{77}{8}x^4+\&c.$, $1+2x-2x^2+4x^3-10x^4+\&c.$

93. 27, 48. 94. 5. 95. (i) $x=9$; (ii) $x=4$, $y=3$; (iii) $x=6$ or $\frac{1}{2}$.

96. $4\frac{7}{12}$ miles an hour; $13\frac{1}{3}$ minutes. 97. x^3.

98. $a^2-2ab+\frac{1}{2}b^2$. 99. x^2-1. 100. $\frac{x^2+b^2}{x}$, $\frac{\sqrt{x}}{\sqrt{x}-\sqrt[4]{a^2y}}$.

101. 4.11. 102. $1-a^2+x^{-2}+3a^{\frac{2}{3}}x^{-\frac{2}{3}}$. 103. 7.

105. 75, 25. 106. $\sqrt{11333311}$ *sept.* $= 2626 = 1000$ *den.*

107. (i) $x=9$; (ii) $x=\pm\frac{1}{2}\sqrt{5}$; (iii) $x=4$, $y=0$, $z=5$.

108. £135, £90. 109. 48. 111. $36x^4-97x^2+36$.

112. $\frac{1}{2}x^2-ax+\frac{1}{3}a^2$. 113. 2.4494, .4082, .8164, 1.2247.

114. $(ab^{-1})^{\frac{1}{2}(m+1)}$, 1. 115. $a^2-4ab-6ac+4b^2+12bc+9c^2$,
$4a^2-2abx-(ac-\frac{1}{4}b^2)x^2+(8ad+\frac{1}{4}bc)x^3-(2bd-\frac{1}{16}c^2)x^4-cdx^5$
$+4d^2x^6$. 116. $\frac{25}{3}\{1-(\frac{2}{5})^n\}$, $8\frac{1}{3}$. 117. $-1\frac{1}{3}$.

118. $1-2x-2x^2-4x^3-10x^4-\&c.$

119. (i) $x=21$; (ii) $x=-3$ or $\frac{5}{7}$; (iii) $x=5$, $y=3$. 120. $3\frac{1}{3}$, 3.

121. $8-12a^{\frac{1}{4}}+18a^{\frac{1}{2}}-27a^{\frac{3}{4}}$. 122. $a^4+\frac{4}{3}a^2bx-\frac{4}{9}(a^2-b^2)x^2+\frac{2}{3}ax^3-\frac{1}{4}x^4$.

123. $\frac{3x^2+4x+2}{4x^2+x+2}$. 124. $a^{10}-a^6x^4-a^4x^6+x^{10}$.

125. 1.2247, $3+\sqrt{3}$. 126. e. 127. 0, $\frac{1}{12}n(7-n)$.

128. A gives 26 guineas and receives 10 crowns.

129. $2(a-x)\sqrt{2ax}$, $\frac{1}{4}\sqrt[4]{a}$. 130. 33 : 238, 1 : 34.
131. (i) $x=10$; (ii) $x=3$, $y=7$; (iii) $x=4$ or -1. 132. 10.
133. With *upper* signs, $16+9=5\times5$; with *lower*, $0+25=5\times5$.
134. x^2+4y. 135. x^2+ax+b. 136. $\frac{3a-2y}{5a+3y}$, $\frac{x(x+5)}{9x^2-x-3}$.
137. $mn(m^2-n^2)(m^2-4n^2)$. 138. $a^{\frac{1}{6}}-a^{\frac{1}{12}}+1$.
139. $\frac{bc}{a^2}(a+b)(a+c)$. 140. 4.8989, .6803, 4.4494, 1.5506, 3.4494.
141. $1\frac{2}{3}$, $2\frac{1}{3}$; $1\frac{2}{7}$, $1\frac{4}{5}$. 143. (i) $x=7$; (ii) $x=4$; (iii) $x=2$ or $\frac{43}{87}$.
142. $1+x+\frac{5}{2}x^2+\frac{15}{2}x^3-\frac{195}{3}x^4+$&c., $1+2x+6x^2+20x^3+70x^4+$&c.
144. £9, 30*s*. 145. $a^2b^2-ab^2x-(a^2+2b^2)x^2+ax^3+2x^4$.
146. $a^2-a^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{15}{4}ax+2a^{\frac{1}{2}}x^{\frac{3}{2}}+4x^2$, $a^4-2a^{\frac{7}{2}}x^{\frac{1}{2}}-\frac{13}{2}a^3x+\frac{23}{2}a^{\frac{5}{2}}x^{\frac{3}{2}}$
$+\frac{289}{16}a^2x^2-23a^{\frac{3}{2}}x^{\frac{5}{2}}-26ax^3+16a^{\frac{1}{2}}x^{\frac{7}{2}}+16x^4$.
147. $x^2-12-16x^{-1}$, a^2+1+a^{-2}. 148. $\frac{a-b+c}{a-2b+3c}$. 149. .2154.
150. $x-2\sqrt{x}+1$. 151. $\sqrt{3}+\sqrt[3]{5}$. 152. $\frac{3}{2}\{1-(\frac{2}{3})^n\}$, $1\frac{1}{2}$.
153. 0, 3, -2. 154. 6. 156. $5\frac{1}{3}$ days, 16 days. 157. $3\frac{19}{40}$.
155. (i) $x=\frac{ab}{a+b}$; (ii) $x=\frac{2}{7}$; (iii) $x=\pm2$, $y=\pm3$.
158. $a^{\frac{4}{3}}-2ab^{\frac{1}{2}}+3a^{\frac{2}{3}}b-2a^{\frac{1}{3}}b^{\frac{3}{2}}+b^2$. 159. $x-5$.
160. $\frac{8(x+6)}{x^4-16}$, $\frac{9}{10}$. 161. $x^4+2x^3-8x^2-6x-1$. 162. 9, 160.
163. $(b+c)^2$. 164. 1147. 165. $1-6x^2+24x^4-80x^6+240x^8-$&c.,
$a^{\frac{3}{2}}\{1+3a^{-1}b+\frac{3}{2}a^{-2}b^2-\frac{1}{2}a^{-3}b^3+\frac{3}{8}a^{-4}b^4+$&c.$\}$.
166. 33233344, 4344 = 1000 *den.*, 244 = 100 *den.*
167. (i) $x=17$; (ii) $x=\frac{ac+b}{a^2+1}$, $y=\frac{c-ab}{a^2+1}$; (iii) $x=\pm6$.
168. £40, £28, or £28, £52, according as *A* had more or less at first than *B*. 169. $\sqrt{\{a^{\frac{3}{2}}+b^{\frac{3}{2}}\}}\times(a^2+b^2)^{\frac{2}{3}}=3.\sqrt[3]{289}=19.834$.
170. $x^{-3}-4x^{-\frac{3}{4}}y^{\frac{3}{3}}+3y^2$. 171. $a^{\frac{pq}{n}}b^{-\frac{pq}{m}}$, $\frac{x(x^2-1)}{2x^2+1}$.
172. $1-\frac{1}{2}ax^{\frac{1}{2}}-2a^2x$. 173. $\frac{ax-b^2}{(x-a)(x-b)}$, $\frac{x+2}{x^2-1}$.
174. 3.8729, 1.2909, .7745, 1.5491, 6.4549.
175. -10, $\frac{1}{4}n(7-3n)$. 176. 15.
177. $1-2x^{\frac{1}{2}}+3x-4x^{\frac{3}{2}}+5x^2-$&c., $1-4x^{\frac{1}{2}}+10x-20x^{\frac{3}{2}}+35x^2-$&c.
178. 12, 16, 18. 179. (i) $\frac{a^2-b^2}{4a-b}$; (ii) 2 or $-1\frac{3}{4}$; (iii) $x=49$, $y=50$.

180. 10 days, $3\frac{1}{5}$ days. 181. $a^3+\frac{1}{2}a^2(2x+y-z)+\frac{1}{4}a(2xy-2xz-yz)$ $-\frac{1}{4}xyz$, which becomes $a^3+3a^2b+3ab^2+b^3$, by putting $x=b=\frac{1}{2}y=-\frac{1}{2}z$, or $x=b$, $y=2b$, $z=-2b$.

182. $5x^2+\frac{7}{2}a^{-\frac{1}{3}}x^{\frac{3}{2}}-\frac{107}{12}a^{-\frac{2}{3}}x+\frac{5}{6}a^{-1}x^{\frac{1}{2}}+\frac{7}{6}a^{-\frac{4}{3}}$. 183. $x+1$.

184. $\dfrac{x^2+2}{x^2+x-5}$ 185. $a^{\frac{1}{3}}x(a^2x^6-1)$. 186. $ab^{-1}-\frac{1}{2}a^{-1}b+1$.

187. $\dfrac{x^2}{(x^2+1)(x-1)^2}$; 188. 720, $\frac{1}{2}(1+\sqrt{7})$. 189. $4\frac{122}{243}$, $\frac{9}{2}\{1+(-\frac{1}{3}{}^n\}$.

190. $7h\ 37'\ 12''$. 191. (i) $1\frac{2}{5}$; (ii) $x=4$, $y=5$; (iii) $4\frac{1}{2}$. 192. 10.

193. $\frac{7}{12}$. 194. $x^{2n}+2$. 195. $5+2\sqrt{6}$, $\sqrt{6}$, $6(5+2\sqrt{6})$.

196. $\left(\dfrac{m+2a}{m+a}\right)^2$, $\dfrac{a^2+ax+x^2}{a^4-x^4}$. 197. $12x^4-2x^3-11x^2+1$.

198. $\frac{9}{5}\{1-(-\frac{2}{3})^n\}$, $1\frac{4}{5}$. 199. n^2. 200. $\pm\frac{3}{5}a$. 201. 15.

202. $1-6x^{\frac{1}{3}}+21x^{\frac{2}{3}}-56x+126x^{\frac{4}{3}}-$&c., $1-3x^{\frac{1}{3}}+6x^{\frac{2}{3}}-10x+15x^{\frac{4}{3}}-$&c.

203. (i) $\frac{2}{5}$; (ii) a^{-1} or b^{-1}; (iii) $y=3$, $y=1$, or $x=\frac{3}{4}$, $y=\frac{1}{4}$.

204. £800, 0. 206. px^2+qx-r. 207. $a-a^{-1}+4$.

208. $\dfrac{x^2+(a^2-a^{\frac{1}{2}})x-a^{\frac{5}{2}}}{x-a}$. 209. $\dfrac{1}{(x^2+1)(x^3+1)}$. 210. 7.0102,202.

211. $\frac{1}{15}$, 20. 212. $(ab)^{\frac{5}{12}}$. 213. 4 yds, 5 yds. 214. 63361, 236, 34.

215. (i) 4 or $1\frac{5}{8}$; (ii) $x=-5$, $y=5$; (iii) $x=\dfrac{a^2}{\sqrt{a^2+b^2}}$, $y=\dfrac{b^2}{\sqrt{a^2+b^2}}$.

216. $\dfrac{mn}{m+n}$ days, $\dfrac{m^2}{n}$ days. 217. $2(n+4)$.

218. $a^{\frac{2}{3}}+b^{4}+c^2-a^{\frac{1}{3}}b^{\frac{2}{3}}-a^{\frac{1}{3}}c-b^{\frac{2}{3}}c$. 219. 2. 220. x^4-y^4.

221. 3.1622, .12649, 2.1081, 1.5811, 4.4414, .31622.

222. $\frac{2}{3}\{1-(\frac{3}{4})^n\}$, $2\frac{2}{3}$. 223. $a^{-3}\{1-3a^{-2}x^2+\frac{15}{2}a^{-4}x^4-\frac{35}{2}a^{-6}x^6$ $+\frac{315}{8}a^{-8}x^8-$&c.$\}$, $\frac{1}{4}a^{-2}(1+3a^{-1}x+\frac{27}{4}a^{-2}x^2+\frac{27}{2}a^{-3}x^3$ $+\frac{405}{16}a^{-4}x^4-$&c.$\}$

224. £5825 8*s* $5\frac{1}{4}d$. 225. 0, -1, 2. 226. 20, 5.

227. (i) $x=a^{-1}$ or $-\frac{1}{2}$; (ii) $x=\dfrac{b}{a^2-ab+b^2}$, $y=\dfrac{a}{a^2-ab+b^2}$; (iii) $x=3$ or $\frac{1}{5}$. 228. $\dfrac{mn(m+n)}{m^2+4mn+n^2}$ days. 229. $2y^2-ay-\frac{1}{2}a^2$.

230. $x^3+a^2x^{\frac{3}{2}}+a^4$, $x^3+2ax^{9}+a^2x^{\frac{3}{2}}-a^4$, $x^3-a^2x^{\frac{3}{2}}-2a^3x^{\frac{3}{4}}-a^4$.

231. a^2-b^2. 232. $x^6-2x^5+x^4-x^2+2x-1$. 233. $\frac{1}{10}$, $\dfrac{\sqrt[3]{x^2+a}}{\sqrt[3]{x^2-a}}$.

234. $\frac{1}{2}x^{\frac{5}{3}}-5y^{\frac{4}{5}}$. 235. 88, $\frac{5}{2}\{1-(\frac{2}{3})^n\}$, $2\frac{1}{2}$, $1\frac{1}{7}$, $1\frac{1}{3}$, $1\frac{3}{5}$.

236. $2a \sim b$. 237. 12, 4, 18 miles. 240. $\dfrac{m^3+n^3}{4mn}$ days.

239. (i) $x = -6\frac{1}{2}$; (ii) $x = \dfrac{m^2-n^2}{am-bn}$, $y = \dfrac{m^2-n^2}{bm-an}$; (iii) $x=10$, $y=7$, $z=3$.

241. $x^{\frac{3}{2}} - xy^{-\frac{3}{4}} + x^{\frac{1}{2}}y^{-\frac{3}{2}} - y^{-\frac{9}{4}}$, $x^2 - (a+b)\,x + ab$.

242. 36, 125. 243. $5x + 4$. 244. $\dfrac{1+b}{ab^2}$, $\frac{1}{4}$.

245. .8164, 1.6329, 2.0412, .1010, 3.2549. 247. $3\frac{1}{3}$, $3\frac{5}{6}$, $4\frac{1}{3}$, &c.

248. 7. 249. 720. 250. 248664*et*69, 54373.

251. (i) $x = 17$; (ii) $x = \dfrac{a^2+ab+b^2}{a+b}$, $y = \dfrac{ab}{a+b}$; (iii) $x = \frac{4}{13}$.

252. $\dfrac{m(m+n)}{3m+n}$ days, $\dfrac{mn(m+n)}{m^2+2mn-n^2}$ days. 253. 140, $\frac{281}{1260}$.

254. $x^{-\frac{4}{3}} - x^{-\frac{2}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$, $x^{\frac{1}{2}} - (m-1)\,a^{\frac{1}{2}}x^{\frac{1}{4}} + a$. 256. $1\frac{1}{3}$, $\dfrac{2a-b}{a^2-1}$.

255. $18x^4 - 45x^3 + 37x^2 - 19x + 6$. 257. $\dfrac{2ac^2}{b(b^2-c^2)}$, $\dfrac{1+x^2}{\sqrt{(1-x^2)^3}}$.

258. 24 miles, $\frac{1}{2}$ hr. 259. $a^2b^{\frac{4}{3}}$. 260. 3. 261. $a + 2x : a + 3x$.

262. 7425. 263. (i) $x = 4$ or $1\frac{2}{5}$; (ii) $x = 10$, $y = -3$, $z = 4$.

264. 8*s* 4*d*. 265. $\frac{25}{4} - \frac{169}{36} = \frac{14}{9} = \frac{8}{3} \times \frac{7}{12}$. 266. $a^{\frac{1}{2}}x + 2$.

267. $a^{\frac{1}{2}} + 3a^{-\frac{1}{2}}x - \frac{9}{2}a^{-\frac{3}{2}}x^2 + \frac{27}{2}a^{-\frac{5}{2}}x^3 - \frac{405}{8}a^{-\frac{7}{2}}x^4 +$ &c., $a + 6x$.

268. a. 269. $\dfrac{1}{(x+1)(x+2)(x+3)}$. 270. £2 8*s*.

271. $n = 10$ or 12, $l = 3$ or -1. 272. 56, 44.

273. $\frac{1}{6}n(n-1)(n-2)$. 274. $1111 \times 10001 = 11111111 = 21845$ *den.*

275. (i) $x = 6\frac{1}{2}$; (ii) $x = \dfrac{b-1}{ab-\frac{1}{2}(a+b)}$, $y = \dfrac{a-1}{ab-\frac{1}{2}(a+b)}$; (iii) $x=4$, $y=3$.

276. $\dfrac{(p-1)\,mn}{np-m}$ days. 277. 0. 279. $\dfrac{3a^{-1}x-4}{a^{-2}x^2-11a^{-1}x+21}$.

278. $\frac{1}{8} \times 64 + 1 = 9 = (\frac{1}{2} \times 4 + 1)(\frac{1}{4} \times 16 - \frac{1}{2} \times 4 \times 1 + 1)$.

280. $a^6x^{10} - a^3x^6 - a^2x^4 + 1$. 281. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 2$. 282. $3\sqrt[3]{5}$, $\frac{29}{2}\sqrt{3}$.

285. 30. 286. 15. 288. 4, 59, 55. 289. $12abc$.

287. (i) $x = 1\frac{1}{3}$; (ii) $x = -4\frac{1}{2}$; (iii) $x = 4$, $y = 3$, or $x = 3$, $y = 4$.

290. $(-\frac{1}{288} + \frac{3}{16} - \frac{1}{144}) - (\frac{7}{288} + \frac{5}{288} - \frac{35}{96}) = \frac{1}{2} = 12\,(-\frac{1}{2} \times \frac{1}{3} \times -\frac{1}{4})$.

291. $1 - \frac{1}{64}a^{-3}x^6$. 292. 84. 293. $2\frac{3}{4}$, .25298, $5 - \sqrt{6}$.

294. $(a^2-b^2)^3$. 296. $\dfrac{n+1}{2n}\,a$, $\dfrac{n^n-a^n}{n(n-a)\,a^{n-2}}$. 297. 10, 20.

299. (i) $x = acb^{-1}$; (ii) $x = \frac{9}{10}a$; (iii) $x = 1$, $y = -1$, or $x = -1\frac{2}{5}$, $y = \frac{3}{5}$.

298. 40320. 300. $\frac{6}{7}$ hr. 301. $3x^{\frac{1}{2}} - 2x^{\frac{1}{4}}y^{-\frac{1}{3}} + 4y^{-\frac{2}{3}}$

302. 2. 303. 137641, $x-2-x^{-1}$. 304. $b-c$.

305. $\sqrt{2}$, $\sqrt{5}+\sqrt{2}$. 306. $\frac{4x^2y^2}{x^4-y^4}$. 307. 1 hr $5\frac{5}{11}'$.

308. $\frac{1}{27}a^2\{1+2a^{-\frac{1}{3}}x^{\frac{1}{3}}+\frac{8}{3}a^{-\frac{2}{3}}x^{\frac{2}{3}}+\frac{80}{27}a^{-1}x+\frac{80}{27}a^{-\frac{4}{3}}x^{\frac{4}{3}}+\text{\&c.}\}$.

309. $1\frac{5}{6}$, $\frac{3}{16}n(n+1)$; $\frac{3}{4}$, $\frac{6}{7}$, 1, $1\frac{1}{5}$. 310. 6.

311. (i) $x=100$ or -10; (ii) $x=\frac{c}{a+b}$, $y=-\frac{c}{a+b}$; (iii) $x=\frac{1-b}{1-ab}$, $y=\frac{a-1}{1-ab}$. 312. $\frac{4}{9}$ hr. 314. $1\frac{1}{3}$, b or b^{-1}.

313. $a+a^{\frac{1}{2}}x^{\frac{1}{2}}+x$, $x-2a^{\frac{3}{4}}x^{\frac{1}{4}}-a^{\frac{1}{2}}x^{\frac{1}{2}}-a$.

315. $8a^{\frac{9}{4}}x^{\frac{3}{2}}-4a^{\frac{3}{2}}xy^{\frac{1}{4}}+2a^{\frac{3}{4}}x^{\frac{1}{2}}y^{\frac{1}{2}}-y^{\frac{3}{4}}$. 316. $\frac{1}{2}(\sqrt{5}-2)$. $\frac{2}{7}xy^{-1}+\frac{3}{4}x^{-1}y-5$.

317. $a^{\frac{1}{2}}\{1+\frac{1}{2}a^{-1}x+\frac{3}{8}a^{-2}x^2+\frac{5}{16}a^{-3}x^3+\frac{35}{128}a^{-4}x^4+\text{\&c.}\}$,
$x^{\frac{1}{3}}\{1-\frac{1}{3}a^{-1}x+\frac{2}{9}a^{-2}x^2-\frac{14}{81}a^{-3}x^3+\frac{35}{243}a^{-4}x^4-\text{\&c.}\}$. 318. 91.

319. $an^{-1}-n$, $a-\frac{1}{2}n(n+1)$. 321. £4 16*s*. 322. 63, £62 8*s*.

323. (i) $x=\frac{2ab}{a+b}$; (ii) $x=2$; (iii) $x=\frac{a^2+bc}{ac}$, $y=\frac{a^2+bc}{c^2}$.

324. 13, 12. 325. $1-x+\frac{11}{12}x^2-\frac{1}{3}x^3+\frac{1}{9}x^4$, $1-2x+\frac{17}{6}x^2-\frac{5}{2}x^3+\frac{83}{48}x^4-\frac{5}{6}x^5+\frac{17}{54}x^6-\frac{3}{27}x^7+\frac{1}{81}x^8$. 326. $2x^3y^{-3}-3x^4y$.

327. 2.64575, .37796, 1.32287, .88191, 1.47683. 328. n^{-1}.

329. $2a\sqrt[6]{a-3x}$, $\frac{2}{3}$, $\sqrt[6]{x^{-1}y}$. 330. $\frac{(x-1)^2}{x^3(x^2+1)^2}$.

331. $1+2xy-3xy^{-1}$. 332. 333. 7. 334. 27907200.

335. (i) $x=\frac{a+b}{c}$ or -1; (ii) $x=2$, $y=1$, $z=0$; but indeterminate, if $2m=n+p$. 336. 5 miles an hour. 337. c or c^{-1}.

338. $\frac{x\sqrt[3]{a}(x\sqrt[3]{a}-1)}{2x\sqrt[3]{a}-1}$. 339. $4\frac{99}{560}$. 340. $\frac{3a}{a+b}$.

341. $4(abc)^{\frac{6}{5}}$. 342. 3.71, $1-2x+3x^2$. 343. 18*s* 4*d*.

345. $(2a)^{\frac{5}{3}}\{1-\frac{5}{2}a^{-\frac{1}{2}}x^{\frac{1}{2}}+\frac{5}{4}a^{-1}x+\frac{5}{24}a^{-\frac{3}{2}}x^{\frac{3}{2}}+\frac{5}{48}a^{-2}x^2+\text{\&c.}\}$,
$(3a)^{-\frac{5}{2}}\{1+\frac{5}{3}a^{-\frac{1}{3}}x^{\frac{1}{3}}+\frac{35}{18}a^{-\frac{2}{3}}x^{\frac{2}{3}}+\frac{35}{18}a^{-1}x+\frac{385}{216}a^{-\frac{4}{3}}x^{\frac{4}{3}}+\text{\&c.}\}$.

346. 2118760, 17296. 347. (i) $x=a\frac{a^2-b^2}{a^2+b^2}$; (ii) $x=1$, $y=4$, $z=27$; (iii) $x=\frac{(a^2+b^2)c}{a^2-b^2}$, $y=\frac{(a^2+b^2)c}{2ab}$. 348. 2 hrs. $21\frac{2}{3}'$.

349. $(x^2+x+1)a-(x+1)$. 350. $a^{-\frac{1}{3}}-a^{\frac{1}{6}}c^{\frac{1}{3}}$. 351. 0.

352. $\frac{3}{2}x^{\frac{3}{2}}-\frac{5}{3}xy^{\frac{1}{2}}+\frac{2}{5}x^{\frac{1}{2}}y$. 353. $\frac{x^3-2x^2+3x-4}{x^4-x^3+x^2-x+1}$.

354. $\frac{27}{2}\{1-(\frac{8}{9})^n\}$, $13\frac{1}{3}$. 355. 76. 356. 9 hrs.

357. $(2a)^{-\frac{5}{3}}\{1+\frac{5}{2}a^{-\frac{1}{2}}x^{\frac{1}{2}}+5a^{-1}x+\frac{55}{6}a^{-\frac{3}{2}}x^{\frac{3}{2}}+\frac{385}{24}a^{-2}x^2+\&c.\}$,
$(3a)^{\frac{5}{2}}\{1-\frac{5}{3}a^{-\frac{1}{3}}x^{\frac{1}{3}}+\frac{5}{6}a^{-\frac{2}{3}}x^{\frac{2}{3}}-\frac{5}{54}a^{-1}x-\frac{5}{648}a^{-\frac{4}{3}}x^{\frac{4}{3}}-\&c.\}$.

358. 4; 1023, 256. 360. $2\frac{2}{9}$ hrs. 361. $x^{\frac{1}{3}}y^{\frac{1}{4}}-x^{\frac{2}{3}}y^{\frac{1}{3}}z$.

359. (i) $\pm\sqrt{\frac{b(bc-2a^2)}{c}}$; (ii) $\frac{a^2\pm\sqrt{a^4+b^2c^2}}{c}$; (iii) $x=\frac{a^2}{a-b}$, $y=\frac{b^2}{b-a}$

363. $x^{\frac{1}{2}}-x^{\frac{1}{4}}+1$. 364. $\sqrt[3]{9}$, $\sqrt{2a}$. 365. 1. 366. 24*s*, 16*s*.

367. $a^{\frac{1}{3}}\{1+2a^{-1}x-4a^{-2}x^2+\frac{40}{3}a^{-3}x^3-\frac{160}{3}a^{-4}x^4+\&c.\}$,
$a^{\frac{2}{3}}\{1+4a^{-1}x-4a^{-2}x^2+\frac{32}{3}a^{-3}x^3-\frac{112}{3}a^{-4}x^4+\&c.\}$.

369. n^3. 370. 1023252 *sen.* = 24e28 *duod.* = 50,000 sq. ft.

371. (i) $x=\frac{b}{a}(a-b+c)$; (ii) $x=a$ or $\frac{b}{na}$; (iii) $x=\pm a\frac{m+n}{\sqrt{m^2+n^2}}$.
$y=\pm a\frac{m-n}{\sqrt{m^2+n^2}}$. 372. $1\frac{1}{2}$h, 1h, 3h. 373. 0. 375. $m^{\frac{3}{2}}-m-m^{\frac{1}{2}}-1$.

374. $mx^3-ax^2-a^2x-(m-2)a^3$. 376. $\pm\sqrt{xy}$. 377. 45. 378. 100, 4.

379. $3\frac{3}{4}$ hrs. 381. 15, 6. 382. 1, 7, 12, or 2, 4, 14, or 3, 1, 16.

383. (i) $x=11$; (ii) $x=\frac{c^2-ab}{a+b}$; (iii) $x=\pm a, y=\mp b$. 384. £740.

385. $(3a-2b)x^2+(a^2-b^2)x$. 387. $\frac{1}{3}, \frac{1}{4}$. 388. $\frac{(a-b)^3}{(x+a)^2(x+b)^2}$.

386. $x^6+(p+1)x^5-(p^2+p+1)x^4-(p+1)(p^2+1)x^3+(p^2+p+1)px^2$
$+(p+1)p^2x-p^3$. 389. $a^{\sqrt{2}}-a^{-\sqrt{2}}$, $a\sqrt{cb^{-1}}-\sqrt{cf}$.

390. $\frac{1}{12}n(3n+1)$. 391. 2.549038 cub. ft. 392. 12 hr $32\frac{8'}{11}$.

393. 2. 394. 5880, 5880, 1960. 396. 2 gals, 14 gals.

395. (i) $x=\frac{a(a+2b)}{2b}$; (ii) $x=\frac{1}{2}(c-a)$; (iii) $x=\pm\frac{24a^2}{25b}$.

398. $x+a$. 399. $x+6$. 400. $x^{\frac{1}{3}}+x^{\frac{1}{5}}+1-x^{-\frac{1}{6}}$.

401. $(a+b)^{\frac{m}{n}+\frac{n}{m}}$. 402. $2x^{-1}$. 403. $5(a-x)^2, n\{(a-x)^2-(n-1)ax\}$.

404. $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(1\pm\sqrt{5})$. 405. .985185312. 406. 215.

407. (i) 5 or -1; (ii) ca^{-1} or cb^{-1}; (iii) $x=\pm\frac{ab^2}{\sqrt{a^4+b^4}}$, $y=\pm\frac{a^2b}{\sqrt{a^4+b^4}}$

408. 4. 410. $2x+3y-z$. 411. 293.0387. 412. $1-\frac{3}{4}x^{\frac{1}{2}}+x^{\frac{1}{3}}$.

414. 0, 1. 416. 15*d*. 418. 23. 420. 256 sq. yds.

419. (i) 5; (ii) $\frac{1}{2}a$ or $\frac{3}{4}a$; (iii) $x=0, y=b, z=c$, or $x=2a$, $y=-b$, $z=-c$.

512.18